Trapping of gravity-capillary water waves by submerged obstacles.

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1. Introduction

Free oscillations with finite energy in the linearised water-wave problem, or trapped modes, have been extensively studied in recent years. Much of the work in this area uses an inverse procedure to construct particular surface-piercing or submerged trapping structures in both two and three dimensions. Initially, gravity was taken to be the sole restoring force, but recently surface-tension effects have been included within the inverse procedure [1]. An alternative method is to specify a class of bodies and to vary parameters until a geometry that supports trapped modes is obtained; this direct method has been used to demonstrate that pairs of submerged ellipses can support trapped modes [4]. The present work describes a criterion, that accounts for surface tension, for the existence of trapped modes supported by given submerged bodies. The method is applied to pairs of ellipses and numerical results are used to demonstrate the effects of surface tension.

2. Statement of the problem

We consider the two-dimensional linear problem which describes interaction between an ideal unbounded fluid W, and bodies B located under the free surface F of the fluid. In particular, the problem can be radiation of waves by forced motion of rigid bodies or diffraction of waves by fixed bodies. The problems appear within the framework of the surface wave theory under the assumptions that the motion is harmonic in time, irrotational and the oscillations have small amplitudes.

We shall use a Cartesian coordinate system (x, y), such that x is a horizontal coordinate and y is a vertical one directed upwards. The motion of the fluid is described by a velocity potential $u(x, y)$ satisfying the following set of conditions:

$$\nabla^2 u = 0 \quad \text{in} \quad W = \mathbb{R}^2_- \setminus B, \quad \mathbb{R}^2_- = \{y < 0\}, \quad (1)$$

$$\partial_y u - \nu u - \beta\partial_x^2 \partial_y u = 0 \quad \text{on} \quad F = \{y = 0\}, \quad (2)$$

$$\partial_n u = f \quad \text{on} \quad S = \partial B, \quad (3)$$

$$\partial_{|x|} u - \mathrm{i}k_0 u = o(1) \quad \text{as} \quad |x| \to \infty, \quad (4)$$

$$\sup_W |\nabla u| < \infty. \quad (5)$$

Here we suppose that $f \in C(S)$, S belongs to the Hölder space $C^{1,\alpha}$ $(0 < \alpha < 1)$ and $\beta = T/(\rho g)$, where T is the surface tension of the fluid, ρ is the fluid density and g is the acceleration due to gravity. Further, k_0 in the radiation condition (4) is the (unique) real positive root of the dispersion relation $\beta k_0^3 + k_0 - \nu = 0$.

Further it will be convenient to use the dimensionless parameter $s = Tk_0^2/(\rho g)$, which gives a measure of the relative importance of surface tension and gravity. Then, with account of the dispersion relation we have $k_0 = \nu/(1 + s)$ and $\beta = s(1 + s)^2\nu^{-2}$.

We shall use Green's function $G(x, y, \xi, \eta)$ for the problem satisfying as a function of the first two arguments the conditions (2), (4), (5) (where supremum is taken over $\mathbb{R}^2_-$ with a vicinity of the point (ξ, η) excluded) and the condition $\nabla^2_{x,y} G(x, y, \xi, \eta) = -\delta(x - \xi)\delta(y - \eta)$, where $y, \eta < 0$ and δ is Dirac's delta-function.

Using the expressions given in [1], we write

$$G(z, \zeta) = -\frac{1}{2\pi}\log\frac{r_-}{r_+} + \frac{1}{\pi(3s + 1)}\sum_{\pm} g_\pm(z, \zeta)$$
$$+ \frac{1 + s}{\pi(3s + 1)}\big[g_0(z, \zeta) + \mathrm{i}\pi \mathrm{e}^{k_0(y+\eta)}\cos k_0(x - \xi)\big]. \quad (6)$$

Here $r_\pm = \sqrt{(x - \xi)^2 + (y \pm \eta)^2}$, $z = x + \mathrm{i}y$, $\zeta = \xi + \mathrm{i}\eta$,

$$g_0(z, \zeta) = \int_0^\infty \frac{\mathrm{e}^{\mu(y+\eta)}\cos\mu(x - \xi)}{\mu - k_0}\,\mathrm{d}\mu,$$

$$g_\pm(z, \zeta) = \mathrm{Re}\bigg\{A_\pm \int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}\mu(z-\bar{\zeta})}}{\mu - a_\pm}\mathrm{d}\mu\bigg\},$$

where

$$a_\pm = k_0\Big(\pm\mathrm{i}\sqrt{\frac{3}{4} + \frac{1}{s}} - \frac{1}{2}\Big), \quad A_\pm = s \pm \frac{\mathrm{i}}{2\sqrt{\frac{3}{4} + \frac{1}{s}}}.$$

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3. Integral equations of potential theory

Following the usual scheme of potential theory (see e.g. [2, ch. 2.1]) we shall seek solutions to the problem in the form of a single layer potential

$$u(z) = \big(V\mu\big)(z), \quad z \in W, \tag{7}$$

where

$$\big(V\mu\big)(z) = \int_S \mu(\zeta)\, G(z,\zeta)\, \mathrm{d}s_\zeta, \tag{8}$$

and μ is some unknown density belonging to $C(S)$. Properties of the single layer potential are described in detail in [2, § 2.1.1.1]. (The arguments are valid in the present case in view of (6), where the Green's function is written as a sum of $-(2\pi)^{-1}\log r_-$ and some smooth in $z, \zeta \in \mathbb{R}^2_-$ function.)

The potential (7) satisfies conditions (1), (2), (4), (5) and the condition (3) leads us to the boundary integral equation

$$-\mu(z) + \big(T\mu\big)(z) = 2\, f(z), \quad z \in S, \tag{9}$$

where

$$\big(T\mu\big)(z) = 2\int_S \mu(\zeta)\, \partial_{n(z)} G(z,\zeta)\, \mathrm{d}s_\zeta.$$

Under the assumption $S \in C^{1,\alpha}$ the operator is compact in $L^2(S)$ and the equation (9) is Fredholm's one. We emphasize that if the equation (9) is solvable in $L^2(S)$ and $f \in C(S)$, the solution μ belongs to $C(S)$.

The adjoint operator T^* appears in the integral equation of the direct method. To obtain the equation we consider Green's identity (we omit derivation of the identity which is fairly standard)

$$u(z) = \int_S \Big[u(\zeta)\, \partial_{n(\zeta)} G(z,\zeta) - \partial_n u(\zeta)\, G(z,\zeta)\Big]\, \mathrm{d}s_\zeta \tag{10}$$

and move the point z onto the contour S. By using the jump relationship for the double layer potentials we arrive at the equation

$$-u(z) + \big(\overline{T^*}u\big)(P) = 2\,\big(Vf\big)(P), \tag{11}$$

where $\big(\overline{T^*}u\big)(z) = 2\int_S u(\zeta)\, \partial_{n(\zeta)} G(\zeta,z)\, \mathrm{d}s_\zeta$.

The arguments applied in [2, § 2.1] for investigation of the integral equations for the water wave problem without surface tension effects can be repeated literally for the problem (1)–(5) and integral equations (9), (11). In this way we find that the problem (1)–(5) is uniquely solvable if and only if the homogeneous boundary integral equations on S

$$-\mu + T\mu = 0, \quad -u + T^*u = 0, \tag{12}$$

have only the trivial solution. Otherwise the equations and the homogeneous problem have the same number of linearly independent solutions and solutions to the second equation (12) are traces of solutions to the homogeneous problem (1)–(5) which in their turn are given by $V\mu$ from solutions to the first equation (12).

We shall use the real and imaginary parts of the operator T^* defined by

$$\big(T_r^*u\big)(z) = 2\int_S u(\zeta)\, \partial_{n(\zeta)} \operatorname{Re} G(\zeta,z)\, \mathrm{d}s_\zeta,$$
$$\big(T_i^*u\big)(z) = 2\int_S u(\zeta)\, \partial_{n(\zeta)} \operatorname{Im} G(\zeta,z)\, \mathrm{d}s_\zeta.$$

It is easy to note that if a function $u : S \mapsto \mathbb{C}$ satisfies equations

$$-u + T_r^*u = 0, \tag{13}$$

$$T_i^*u = 0, \tag{14}$$

then the equations also hold for $\operatorname{Re} u$ and $\operatorname{Im} u$. At the same time, for any real-valued function v we have $(T_r^* - \mathrm{i}T_i^*)v = T^*v$. This means that each of the functions $\operatorname{Re} u$ and $\operatorname{Im} u$ satisfy the second equation (12) and it is also true for u.

Let us now show that any solution to the second equation (12) satisfies the system (13), (14). For this we shall need some asymptotic analysis for a solution to the problem (1)–(5). The Green identity (10) and asymptotic representations of Green's function give us the following asymptotics as $|z| \to \infty$ and $\pm x > 0$:

$$u(z) = C_\pm \mathrm{e}^{k_0 y} \mathrm{e}^{\mathrm{i}k_0|x|} + \varphi_\pm(z), \tag{15}$$

where $C_\pm$ are some complex coefficients and $\partial_x^n \partial_y^m \varphi_\pm = O(|z^{-1}|)$ as $|z| \to \infty$ for any $n, m \geqslant 0$.

Let u be a solution to the homogeneous problem (1)–(5). Consider a domain $W_a = W \cap \{|x| < a\}$ and apply Green's formula over the domain for u and $\overline{u}$. We have

$$0 = \int_{W_a} \big[\overline{u}\nabla^2 u - u\nabla^2\overline{u}\big]\, \mathrm{d}x\mathrm{d}y = \int_{\partial W_a} \big[u\, \partial_n \overline{u} - \overline{u}\, \partial_n u\big]\, \mathrm{d}s, \tag{16}$$

where the normal $\boldsymbol{n}$ is directed to the interior of W_a.

We write $\partial W_a = S \cup F_a \cup L_+ \cup L_-$, where $F_a = F \cap \{|x| < a\}$, $L_\pm = \{(x,y) : x = \pm a,\ y < 0\}$. Taking into account that $\partial_n = -\partial_y$ and $u = \nu^{-1}(u_y - \beta u_{xxy})$ on F_a (by (2)), in view of (15) we find

$$\int_{F_a} \big[u\, \partial_n \overline{u} - \overline{u}\, \partial_n u\big]\, \mathrm{d}x = \frac{\beta}{\nu}\int_{F_a} \big[\overline{u}_y u_{xxy} - u_y \overline{u}_{xxy}\big]\, \mathrm{d}x$$
$$= \frac{\beta}{\nu}\big[\overline{u}_y u_{xy} - u_y \overline{u}_{xy}\big]_{x=-a}^{x=a} = \frac{2\mathrm{i}\beta k_0^3}{\nu}\big(|C_+|^2 + |C_-|^2\big) + o(1) \quad \text{as} \quad a \to \infty.$$

Analogously, we have

$$\int_{L_\pm} \big[u\, \partial_n \overline{u} - \overline{u}\, \partial_n u\big]\, \mathrm{d}y = \pm\int_{-\infty}^{0} \big[\overline{u}u_x - u\overline{u}_x\big]\, \mathrm{d}y$$
$$= \mathrm{i}|C_\pm|^2 + o(1) \quad \text{as} \quad a \to \infty.$$

Finally, in the limit $a \to \infty$ we find from (16) $(1 + 2\beta\nu^{-1}k_0^3)\big(|C_+|^2 + |C_-|^2\big) = 0$ and conclude that a solution to the homogeneous problem (1)–(5) decays to zero at infinity and it is the so-called trapped mode. The latter means that each of the functions $\operatorname{Re} u$, $\operatorname{Im} u$ is a solution to the homogeneous problem, its trace on S is a solution to the second equation (12) and, thus, satisfies the system (13), (14). Finally, we can conclude that the second equation (12) is equivalent to (13), (14).

4. Criterion for uniqueness

We shall use the formalism suggested in [3] and based on symmetrization of the integral equation (13) and the adjoint one $-\mu + T_r\mu = 0$. Applying the operator $(I - T_r^*)$ to the last equation and $(I - T_r)$ to (13) we arrive at

$$-\mu + \mathfrak{T}\mu = 0, \quad \mathfrak{T} = T_r + T_r^* - T_r^*T_r, \tag{17}$$

$$-u + \mathfrak{T}'u = 0, \quad \mathfrak{T}' = T_r + T_r^* - T_rT_r^*, \tag{18}$$

and, obviously, solutions to

$$(I - T_r)\mu = 0, \quad (I - T_r^*)u = 0 \tag{19}$$

satisfy (17) and (18), respectively. It can also be observed that solutions to (17) and (18) satisfy (19). Consider for example (17); it can be written as $(I - T_r^*)(I - T_r)\mu = 0$ and either $(I - T_r)\mu = 0$ or $(I - T_r)\mu = v \neq 0$ and $(I - T_r^*)v = 0$. By Fredholm's alternative the subspace $\operatorname{Ker}(I - T_r^*)$ is orthogonal to $\operatorname{Im}(I - T_r)$. Hence $\langle v, v\rangle = 0$ because v belongs to both subspaces. (We denote by $\langle v, w\rangle$ the scalar product of v and w in $L_2(S)$.)

It is important to note that $\mathfrak{T}$ and $\mathfrak{T}'$ are compact and, unlike T_r, self-adjoint operators with real eigenvalues $\lambda_i \in \sigma(\mathfrak{T})$ and $\lambda_i' \in \sigma(\mathfrak{T}')$. It can be observed that $\langle (I-T_r^*)(I-T_r)v, v\rangle = \langle (I-T_r)v, (I-T_r)v\rangle \geqslant 0$. Thus, $\langle \mathfrak{T}v, v\rangle \leqslant \langle v, v\rangle$ and all eigenvalues of the operator $\mathfrak{T}$ are submitted to the inequality $\lambda_i \leqslant 1$. Analogously, $\lambda_i' \leqslant 1$.

Further we shall use the notation $\lambda_1 = \max\{\lambda_i\}$. It follows from the above that (13) has only the trivial solution if and only if $\lambda_1 < 1$, and non-trivial solutions exist only when $\lambda_1 = 1$.

Let now $\lambda_1 \neq 1$. It can be observed that in this case the operators $\mathfrak{T}$ and $\mathfrak{T}'$ have the same system of eigenvalues $\{\lambda_i\}$ and dimensions N_i and N_i' of eigenspaces $\mathcal{E}_i$ and $\mathcal{E}_i'$ corresponding to any eigenvalue λ_i are equal. Consider an eigenvalue λ_i of the operator $\mathfrak{T}$. From the definition of the eigenvalue and taking into account the definition of $\mathfrak{T}$ for $\mu_i \in \mathcal{E}_i$ we have

$$(I - T_r^*)(I - T_r)\mu_i = (1 - \lambda_i)\mu_i.$$

Applying the operator $(I - T_r)$ from the left to the latter equality we arrive at

$$(I - T_r)(I - T_r^*)(I - T_r)\mu_i = (1 - \lambda_i)(I - T_r)\mu_i,$$

or $(I - \mathfrak{T}')u_i = (1 - \lambda_i)u_i$, where $u_i = (I - T_r)\mu_i$. The latter means that λ_i is an eigenvalue of $\mathfrak{T}'$ and $(I - T_r)\mathcal{E}_i \subset \mathcal{E}_i'$. Since $(I - T_r)$ is bijective on $L^2(S)$, this means that $N \leqslant N'$. Applying now $(I - T_r^*)$ to $\mathcal{E}_i$ we find $(I - T_r^*)\mathcal{E}_i' \subset \mathcal{E}_i$ and $N' \leqslant N$ and, finally, $N = N'$.

Let us denote by $\mu_1^{(i)}$ and $u_1^{(i)}$ $(i = 1, \ldots, N)$ eigenfunctions of $\mathfrak{T}$ and $\mathfrak{T}'$ corresponding to λ_1. We can prove that if for some i and for all $j = 1, \ldots, N$

$$\big\langle (I - T_r^*)u_1^{(i)}, \mu_1^{(j)}\big\rangle = 0, \tag{20}$$

then $\lambda_1 = 1$. Suppose the contrary, i.e. $\lambda_1 \neq 1$. Then the mapping $(I - T_r^*)$ is a bijection between $\operatorname{span}\{u_1^{(1)}, \ldots, u_1^{(N)}\}$ and $\operatorname{span}\{\mu_1^{(1)}, \ldots, \mu_1^{(N)}\}$. This means that we can write

$$(I - T_r^*)u_1^{(i)} = \sum_{k=1}^{N} c_k \mu_1^{(k)}.$$

Then from (20) we obtain

$$\sum_{k=1}^{N} c_k \big\langle \mu_1^{(k)}, \mu_1^{(j)}\big\rangle = 0, \quad j = 1, \ldots, N.$$

Since $\mu_1^{(i)}$ are linearly independent, from the last system of linear equation with Gram matrix it follows that $c_k = 0$, $k = 1, \ldots, N$, and $(I - T_r^*)u_1^{(i)} = 0$, which contradicts the assumption that $\lambda_1 \neq 1$.

Summing up the above arguments we conclude that non-trivial solutions to the homogeneous problem (1)–(5) exist if and only if some eigenfunction $u_1^{(i)}$ satisfies the conditions (20) and the condition (14). In the following section we shall check these conditions numerically.

5. Trapped modes: numerical results

We shall consider a configuration consisting of two equal ellipses with horizontal semi-axis a and vertical semi-axis b, with centres on the depth d and distance between centres $2l$. We shall use the data obtained in the work [4] for this geometry in the water wave problem without surface tension.

In view of (6) for real-valued function u we have

$$\frac{1 + 3s}{1 + s}\,\big(T_i^* u\big)(z) = 2\mathrm{e}^{k_0 y}\cos k_0 x \operatorname{Re}\big\{\Theta(u)\big\} + 2\mathrm{e}^{k_0 y}\sin k_0 x \operatorname{Im}\big\{\Theta(u)\big\},$$

where

$$\Theta(u) := \int_S u(\zeta)\partial_n\big(\mathrm{e}^{k_0\eta}\mathrm{e}^{\mathrm{i}k_0\xi}\big)\,\mathrm{d}s_\zeta.$$

Therefore, a function u satisfies (14) if it satisfies the condition $\Theta(u) = 0$.

Assume that the eigenvalue λ_1 is simple. Then the following conditions (where u_1 and μ_1 are the values defined in the previous section)

$$\Psi := \left\langle (I - T_r^*)u_1, \mu_1 \right\rangle = 0, \quad \Theta(u_1) = 0, \qquad (21)$$

guarantee existence of a trapped mode. When λ_1 is simple and S is symmetric with respect to the y-axis, the function u_1 is either even or odd in x and the second of conditions (21) is reduced to one real condition.

Further we shall show existence of solutions to the system (21) numerically. To make the computation convincing we shall use a scheme (see e.g. [4]) based on consideration of two curves, defined by the equations, in the plane of two parameters (here it will be ν and l). If the curves intersect, then this situation is stable with respect to numerical inaccuracy.

Another important observation is the possibility to define Ψ, Θ as piecewise-analytic in ν functions (basically, even under the normalization $\|u_1\| = \|\mu_1\| = 1$ they are defined up to the sign). This allows us to find zeros of the functionals convincingly.

Computations are carried out for two ellipses when $b/a = 0.08$, $d/a = 0.25$, by using a simple collocation scheme for approximation of the integral operator T_r. Each of the ellipses is split into $N/2$ arcs uniformly in $\theta \in [0, 2\pi]$ in the representation $x(\theta) = \pm l + a\cos\theta$, $y(\theta) = -d + b\sin\theta$. Examples of computation are presented in fig. 1a and 1b, where for $s = 0.002$ and for $s = 0.006$, respectively, we show behaviour of the curves $\Psi = 0$ (I) and $\Theta = 0$ (II) in the plane of parameters l/a, νa (this curves are found for $N = 240$). We also check numerically that $\lambda_1 - \lambda_2$ is separated from zero at the point of intersection of the curves (I) and (II). It is very likely that the simplicity of λ_1 takes place for all values of parameters $(l/a, \nu a)$, shown in fig. 1a and 1b, except the values belonging to the curves (III), where multiplicity of λ_1 is equal to two.

Results for the computation of trapped modes parameters for some values of the parameter s are presented in the adjoint table.

s	νa	l/a
0	0.584...	2.34...
0.001	0.579...	2.38...
0.002	0.574...	2.41...
0.003	0.569...	2.45...
0.004	0.562...	2.50...
0.005	0.554...	2.56...
0.006	0.543...	2.63...

We can observe that the computations are sufficiently strongly influenced by the surface tension effects. With increase of s intersection of the curves $\Psi = 0$ and $\Theta = 0$ becomes less distinct. Hence we were only able to find trapped modes for small

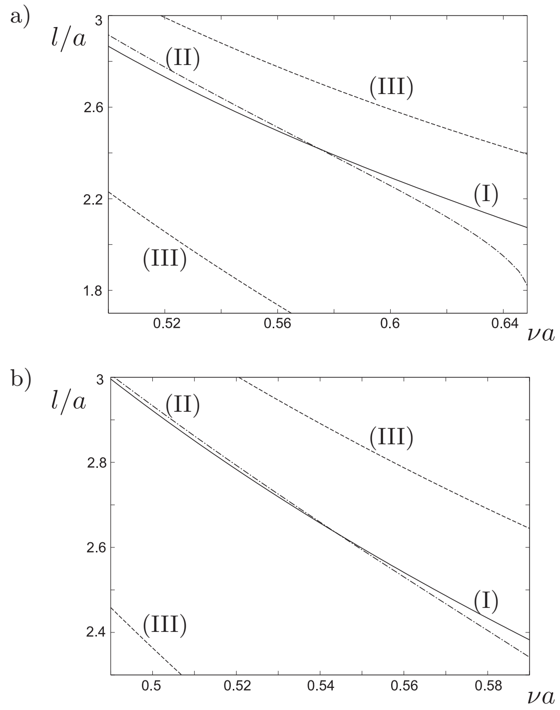

Figure 1: Curves $\Psi = 0$ (solid line, I), $\Theta = 0$ (dash-and-dot line, II) and points of u_1 parity change (dash line, III) for $s = 0.002$ (a) and for $s = 0.006$ (b).

values of the parameter s, which are however sufficiently large from physical point of view. Consider for example the pair water–air, where the coefficient of surface tension T is approximately equal to 0.072 N·m^{-1} and $\rho = 1000$ kg·m^{-3}. Solving the equation $s(1 + s^2) = T(\nu a)^2/(a^2\rho g)$ for $s = 0.006$ and the corresponding value $\nu a = 0.543$ (see the table above) we find $a \approx 19$ mm and smaller values of s correspond to bigger values of the ellipse length.

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