Wave interaction with a submerged horizontal plate/membrane in a channel

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1 INTRODUCTION

Wave scattering by submerged barriers is key for coastal structures like breakwaters. It is well established that submerged plates and membranes reduce visual impact while absorbing wave energy. However, a general model for plates in a channel is still lacking. Channel walls create guided waves and resonances, changing how waves scatter compared to open water. This paper introduces a semi-analytical model based on linear potential flow and Fourier transforms for wave scattering by a submerged rectangular plate or membrane in a channel, covering rigid, flexible, porous, or impermeable types. The Galerkin method with Chebyshev polynomials accurately addresses edge velocity singularities. The model calculates reflection/transmission, forces, and deformation. Its contributions are: (1) a single framework for different plates/membranes, (2) inclusion of channel and edge effects that are often ignored in open-water studies, and (3) discovery of new effects like a force drop near resonances and an optimal width for wave dissipation.

2 MATHEMATICAL MODEL

We study wave scattering by a submerged horizontal rectangular plate/membrane within a channel of width $2w$, as shown in Fig. 1. A Cartesian coordinate system $Oxyz$ is defined with z vertical, the origin at the mean free surface, and the x-axis aligned with wave propagation. The plate/membrane, of length $2a$ and width $2b$, is submerged at depth d. Its centre is offset by distance c from the channel centreline. We assume the fluid is incompressible and

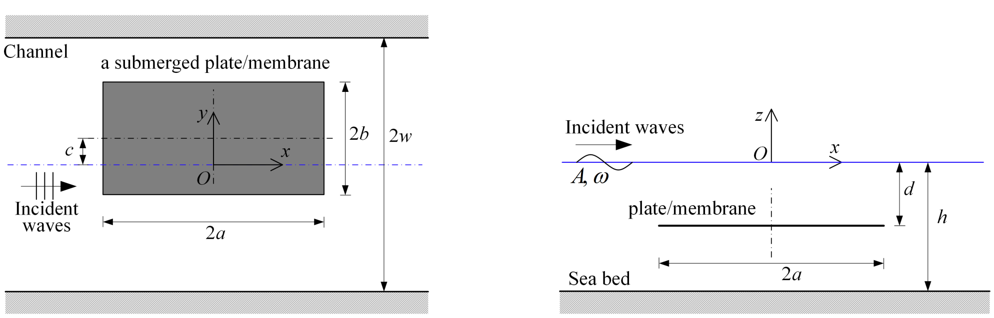

Figure 1: A submerged horizontal plate in a channel: (left) plan view; (right) side view.

inviscid, the flow is irrotational and that the wave amplitude is small, so that linear theory applies. We denote the fluid velocity potential by $\Phi(x, y, z, t)$. It is further assumed that all motion is time-harmonic with angular frequency ω. Thus, we can write

$$\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, y, z)\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{1}$$

where ϕ is the spatial velocity potential independent of time, i.e., t. i is the imaginary unit.

We divide the velocity potential into two components, such that

$$\phi = \phi_I + \phi_D, \tag{2}$$

where ϕ_I denotes the undisturbed incident wave potential $\phi_I = -\mathrm{i}gA\mathrm{e}^{\mathrm{i}kx}\psi(z)/[\omega\cosh(kh)]$, with $\psi(z) = \cosh[k(z+h)]$, and ϕ_D represents the diffracted wave velocity potential. Here k is the wave number, g is the acceleration due to gravity, and A is the incident wave amplitude.

Given that the boundary condition at the channel wall $\partial\phi_D/\partial y = 0$ at $y = \pm w$ must be satisfied, the diffracted velocity potential ϕ_D could be further modally expanded as

$$\phi_D(x, y, z) = \sum_{m=0}^{\infty} \phi_m(x, z)\cos\left[m\pi(y + w)/(2w)\right]. \tag{3}$$

The m-th modal velocity potential ϕ_m satisfies

$$\left[\partial^2/\partial x^2 + \partial^2/\partial z^2 - [m\pi/(2w)]^2\right]\phi_m = 0 \quad \text{in the water}, \tag{4}$$

$$\partial\phi_m/\partial z = K\phi_m, \quad z = 0, \tag{5}$$

$$\partial\phi_m/\partial z = 0, \quad z = -h, \tag{6}$$

where $K = \omega^2/g$. The velocity potential should also satisfy the kinematic boundary condition at the horizontal plate/membrane

$$\frac{\partial\phi_D}{\partial z} + \frac{\partial\phi_I}{\partial z} = \frac{\partial\phi^+}{\partial z} = \frac{\partial\phi^-}{\partial z} = -\mathrm{i}p_0P(x,y) - \mathrm{i}\omega\eta, \text{ at } z = -d,\ |x| < a,\ |y - c| < b, \tag{7}$$

where $P(x, y) = \phi^+ - \phi^-$ denotes the velocity potential jump across the plane $z = -d$. p_0 denotes a complex porosity parameter when the plate/membrane is porous. η represents the deflection of the structure, and $\eta = 0$ for a rigid plate. $P(x, y)$ can be further expanded as

$$P(x, y) = \sum_{m=0}^{\infty} P_m(x)\cos\left[m\pi(y + w)/(2w)\right], \tag{8}$$

where $P_m(x) = \phi_m(x, -d^-) - \phi_m(x, -d^+)$. Clearly, $P(x, y) = 0$ at $|x| > a$ and/or $|y - c| > b$; and $P_m(x) = 0$ at $|x| > a$.

The radiation boundary condition at $x \to \pm\infty$,

$$\phi(x,y,z) \sim \begin{cases} \phi_I(x,z) + \displaystyle\sum_{m=0}^{M} R_m \mathrm{e}^{-\mathrm{i}(\alpha_m - k)x}\cos\left(\frac{m\pi(y+w)}{2w}\right)\phi_I(-x,z), & x \to -\infty \\ \displaystyle\sum_{m=0}^{M} T_m \mathrm{e}^{\mathrm{i}(\alpha_m - k)x}\cos\left(\frac{m\pi(y+w)}{2w}\right)\phi_I(x,z), & x \to +\infty, \end{cases} \tag{9}$$

should also be satisfied, where $M = \lfloor\frac{2kw}{\pi}\rfloor$ and $\alpha_m = \sqrt{k^2 - \left(\frac{m\pi}{2w}\right)^2} \geq 0$ for $0 \leq m \leq M$. R_m and T_m are complex reflection and transmission coefficients of the m-th cosine mode in the $y-$ direction.

For a flexible rectangular plate or membrane, the dynamic boundary condition on the structure

$$\mathrm{i}\omega P(x,y) = \begin{cases} g\left(-\chi\Delta^2 + K\gamma\right)\eta, & \text{for a elastic plate} \\ g\left(\tau\Delta + K\gamma\right)\eta, & \text{for a membrane} \end{cases} \tag{10}$$

should also be satisfied, where Δ denotes the Laplacian operator in the horizontal plane. χ denotes the flexural rigidity of the plate; γ represents the mass per unit area of the plate/membrane. τ represents the effect of equal uniform tension acting along the membrane in all directions. Details of the physical meanings of these parameters are given by [1]. Edge boundary conditions of the plate/membrane, e.g., simply supported, must be considered because they affect the expressions for η derived using the dry mode expansion method.

We define the Fourier transform and its inverse as

$$\bar{\phi}_m(l,z) = \int_{-\infty}^{\infty} \phi_m(x,z)\mathrm{e}^{-\mathrm{i}lx}\mathrm{d}x, \quad \phi_m(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\phi}_m(l,z)\mathrm{e}^{\mathrm{i}lx}\mathrm{d}l, \tag{11}$$

where l is an independent wave number in the spectral domain.

After applying the Fourier transform, following Porter [2], we have

$$\left(\partial^2/\partial z^2 - \gamma_m^2\right)\bar{\phi}_m(l,z) = 0, \tag{12}$$

$$\frac{\partial\bar{\phi}_m(l,z)}{\partial z} = K\bar{\phi}_m(l,z), \quad z = 0, \tag{13}$$

$$\frac{\partial\bar{\phi}_m(l,z)}{\partial z} = 0, \quad z = -h, \tag{14}$$

where $\gamma_m^2 = (m\pi/2w)^2 + l^2$, and the continuous velocity condition at the plane $z = -d$

$$\partial\bar{\phi}_m(l,z)/\partial z\big|_{z=-d^-} = \partial\bar{\phi}_m(l,z)/\partial z\big|_{z=-d^+}, \tag{15}$$

and $P_m(x)$ can be converted into

$$\bar{P}_m(l) = \int_{-\infty}^{\infty} P_m(x)\mathrm{e}^{-\mathrm{i}lx}\mathrm{d}x = \int_{-a}^{a} P_m(x)\mathrm{e}^{-\mathrm{i}lx}\mathrm{d}x = \bar{\phi}_m(l,-d^-) - \bar{\phi}_m(l,-d^+). \tag{16}$$

Solving Eq. (12) in $z \in [-d,0]$ with (13) and in $z \in [-h,-d]$ with (14) and matching across $z = -d$ using Eqs. (15) and (16) gives

$$\bar{\phi}_m(l,z) = \begin{cases} \dfrac{\bar{P}_m(l)\sinh[\gamma_m(h-d)](\gamma_m\cosh(\gamma_m z) + K\sinh(\gamma_m z))}{\gamma_m\sinh(\gamma_m h) - K\cosh(\gamma_m h)}, & z \in [-d,0], \\ \dfrac{\bar{P}_m(l)\cosh[\gamma_m(z+h)](-\gamma_m\sinh(\gamma_m d) + K\cosh(\gamma_m d))}{\gamma_m\sinh(\gamma_m h) - K\cosh(\gamma_m h)}, & z \in [-h,-d]. \end{cases} \tag{17}$$

Taking the inverse Fourier transform of Eq. (17) in $z \in [-d,0]$ gives

$$\phi_m(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{P}_m(l)\sinh[\gamma_m(h-d)](\gamma_m\cosh(\gamma_m z) + K\sinh(\gamma_m z))}{\gamma_m\sinh(\gamma_m h) - K\cosh(\gamma_m h)}\mathrm{e}^{\mathrm{i}lx}\mathrm{d}l. \tag{18}$$

For ϕ_m, when $m \leq M$, poles must occur on the real axis at $l = \pm\alpha_m$ where $\gamma_m = k$. To satisfy the radiation boundary condition that ϕ_D is outgoing, the contour of integration is taken to pass over the pole at $l = -\alpha_m$ and under the pole at $l = \alpha_m$.

Singularities of fluid motion occur at the four edges of the submerged plate/membrane. Following Porter [2], we expand the unknown potential jump function as

$$P(x,y) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_{p,q} w_p\left(\frac{x}{a}\right) w_q\left(\frac{y-c}{b}\right), \quad x \in [-a,a],\ y \in [c-b, c+b] \tag{19}$$

where $a_{p,q}$ are unknown coefficients to be determined, $w_p(t) = \mathrm{i}^p\sqrt{1-t^2}U_p(t)/[(p+1)\pi]$, in which $U_p(t)$ are second-kind Chebychev polynomials with square-root behaviour at $t = \pm 1$.

For a rigid plate, after inserting the expressions of the potential jump Eq. (19) and the diffracted velocity potential in terms of Eq. (18) into the kinematic boundary condition Eq. (7), and making use of the orthogonal properties of Chebychev polynomials and trigonometric functions, the unknown coefficients can be determined. For a flexible plate/membrane, the dry mode expansion of the deflection and the dynamic boundary condition Eq. (10) are employed to predict the response of the structure further.

3 RESULTS

Figure 2 presents the effect of plate length on the hydrodynamics of a simply-supported flexible porous plate. Regardless of plate length, a sharp valley is observed in the frequency response curve of the heave force around $kh = \pi$, corresponding to wave resonance across the channel in the y-direction. The wave power dissipation coefficient η_d usually increases with increasing wave frequency; however, a drop occurs around $kh = \pi$. The longer the plate, the more wave power can be dissipated over the entire range of wave frequencies examined.

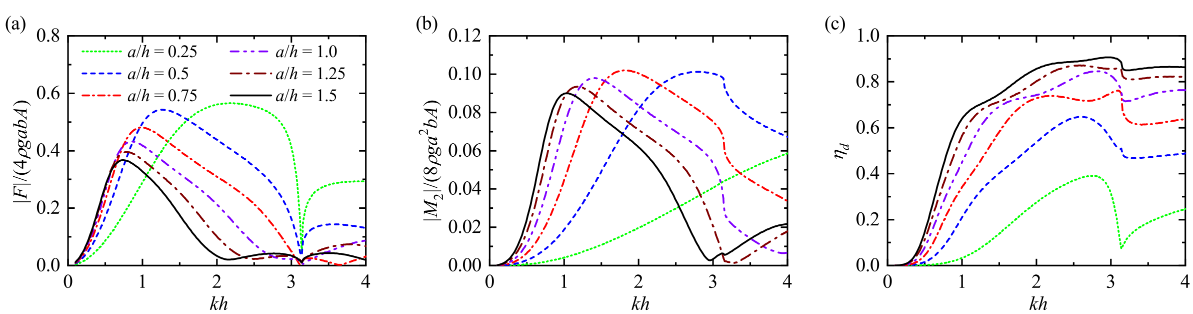

Figure 2: Frequency responses of a flexible porous rectangular plate submerged in a channel for different values of plate length with $\bar{\chi} = \chi/h^4 = 0.01$, $\bar{\gamma} = \gamma/h = 0.01$, $\bar{p} = 2\pi p_0/k = 5.0$, $b/h = 0.5$, $c/h = 0$, $d/h = 0.1$, and $w/h = 1.0$: (a) wave-induced heave force; (b) wave-induced pitch moment; and (c) wave power dissipation coefficient η_d.

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