

Modular Green-Naghdi Theory using B-Splines

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1 INTRODUCTION

Webster et al. [1] proposed a streamlined derivation of high-level Green-Naghdi (HLGN) models, substantially improving the accuracy of dispersive wave calculations. Upon convergence, the HLGN predictions are comparable to exact Euler’s solutions. Consequently, GN models have been widely used in recent years for nonlinear surface and internal waves [2, 3].

Traditional GN models represent the vertical structure of the velocity field using shape functions that typically span the entire water column. While effective in shallow water, this approach becomes inefficient for deep-water conditions where the dynamics are largely confined to a thin near-surface layer, and it is inadequate for steep, nearly breaking, broadband wave fields with small near-surface length scales.

Motivated by recent work on simulation surface wave in a two-layer system [4], we develop a layer-based organization of the GN model in which the flow is constructed from stacked fluid layers, termed the Modular Green-Naghdi (MGN) model. The framework enables locally tailored vertical resolution and basis representation. In particular, the intralayer variation of horizontal velocity is represented using B-spline curves with coefficients that depend continuously on the horizontal coordinates and time. An outline of the corresponding MGN equations is presented in this paper.

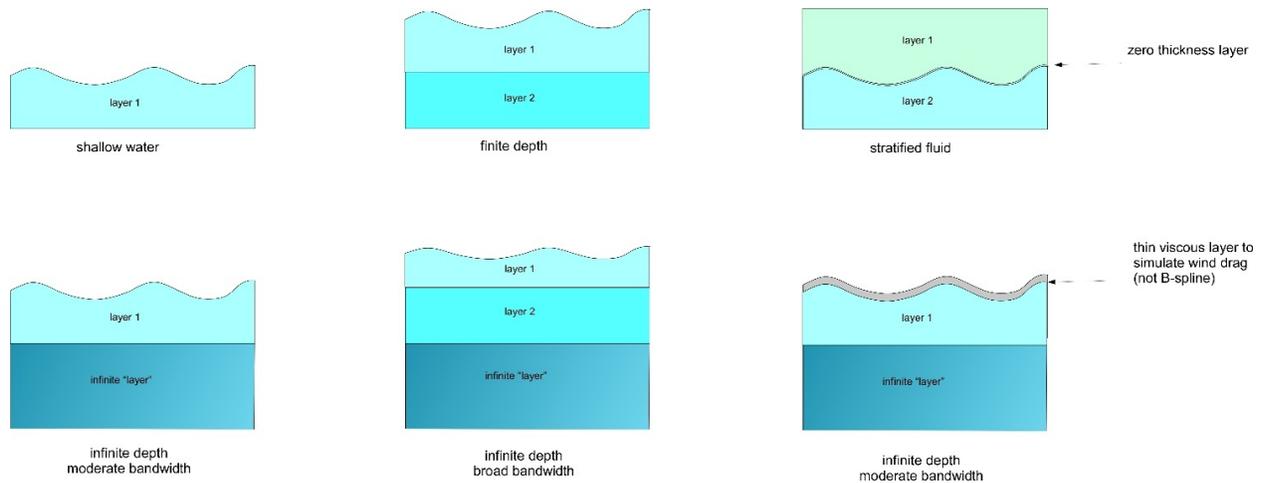


Figure 1: Possible situations for use of Modular Green-Naghdi theory

We have chosen to use layers since with them as stacked modules, since a variety of different problems can be treated (see Fig. 1). In its most general configuration, the layer consists of top and bottom surfaces that can be flat or curved, be moving or fixed, and be material surfaces (no leakage across the boundary) or porous.

Development of a Green-Naghdi method involves two steps: The selection of a kinematic framework for the motion of the fluid, and the use of a weak form of the conservation of momentum, and the imposition of appropriate boundary conditions to develop a model for a specific problem.

2 THE KINEMATIC FRAMEWORK

The kinematic framework is formed so that general problems can be reasonably be represented by the universe of possibilities within this framework. We start with the consideration of a single layer. For reference, we take the horizontal coordinates to be x and y , and the vertical coordinate to be z . With this in mind, we have chosen the “shape functions”, the variation of the horizontal flow variables in the z -direction, to be cubic spline curves determined by the values of the horizontal velocities and their z -derivatives at the bottom surface of the layer, $z = z_\alpha(x, y, t)$, i.e., $u_\alpha(x, y, t)$, $u_{z,\alpha}(x, y, t)$, $v_\alpha(x, y, t)$, $v_{z,\alpha}(x, y, t)$, and the values of the horizontal velocities and z -derivatives at the top surface, $z = z_\beta(x, y, t)$, i.e., $u_\beta(x, y, t)$, $u_{z,\beta}(x, y, t)$, $v_\beta(x, y, t)$, $v_{z,\beta}(x, y, t)$.

This choice of independent variables assures that when layers are stacked, these horizontal velocities will be continuous and smooth across the join. It can be shown that, with the construction described below, that the vertical velocity, $w(x, y, z, t)$ is also continuous and smooth across the join. That is, we model the horizontal velocities by:

$$u(x, y, z, t) = u_\alpha(x, y, t)\sigma_1(z) + u_{z,\alpha}(x, y, t)\sigma_2(z) + u_\beta(x, y, t)\sigma_3(z) + u_{z,\beta}(x, y, t)\sigma_4(z), \quad (1a)$$

$$v(x, y, z, t) = v_\alpha(x, y, t)\sigma_1(z) + v_{z,\alpha}(x, y, t)\sigma_2(z) + v_\beta(x, y, t)\sigma_3(z) + v_{z,\beta}(x, y, t)\sigma_4(z), \quad (1b)$$

where the B-spline primitives are:

$$\sigma_1(z) = 1 - 3\frac{(z - z_\alpha)^2}{(z_\beta - z_\alpha)^2} + 2\frac{(z - z_\alpha)^3}{(z_\beta - z_\alpha)^3}, \quad \sigma_2(z) = (z - z_\alpha) - 2\frac{(z - z_\alpha)^2}{(z_\beta - z_\alpha)} + \frac{(z - z_\alpha)^3}{(z_\beta - z_\alpha)^2}, \quad (2a)$$

$$\sigma_3(z) = 3\frac{(z - z_\alpha)^2}{(z_\beta - z_\alpha)^2} - 2\frac{(z - z_\alpha)^3}{(z_\beta - z_\alpha)^3}, \quad \sigma_4(z) = -\frac{(z - z_\alpha)^2}{(z_\beta - z_\alpha)} + \frac{(z - z_\alpha)^3}{(z_\beta - z_\alpha)^2}. \quad (2b)$$

In the Modular Green-Naghdi model, we consider the fluid to be incompressible and thus the fluid velocities satisfy the continuity condition, $\nabla \cdot \mathbf{u} = 0$, where $\mathbf{u} = [u, v, w]$. Using our model for the horizontal velocities, Eq. (1), we get the following expression for the vertical velocity, w :

$$w(x, y, z, t) = u_\alpha(x, y, t)\delta_1(z) + u_{z,\alpha}(x, y, t)\delta_2(z) + u_\beta(x, y, t)\delta_3(z) + u_{z,\beta}(x, y, t)\delta_4(z) \\ + v_\alpha(x, y, t)\delta_1(z) + v_{z,\alpha}(x, y, t)\delta_2(z) + v_\beta(x, y, t)\delta_3(z) + v_{z,\beta}(x, y, t)\delta_4(z) \\ + w_\alpha(x, y, t), \quad (3a)$$

$$w_\beta(x, y, t) = w_\alpha + \frac{1}{12} [6u_{x,\alpha} + 6u_{x,\beta} - (u_{xz,\alpha} - u_{xz,\beta})(z_\alpha - z_\beta)] (z_\alpha - z_\beta) \\ + \frac{1}{12} [6v_{x,\alpha} + 6v_{x,\beta} - (v_{xz,\alpha} - v_{xz,\beta})(z_\alpha - z_\beta)] (z_\alpha - z_\beta), \quad (3b)$$

where

$$\delta_1(z) = -(z - z_\alpha) + \frac{(z - z_\alpha)^4}{2(z_\beta - z_\alpha)^3} + \frac{(z - z_\alpha)^3}{(z_\beta - z_\alpha)^3}, \quad \delta_3(z) = \frac{(z - z_\alpha)^4}{2(z_\beta - z_\alpha)^3} - \frac{(z - z_\alpha)^3}{(z_\beta - z_\alpha)^2}, \quad (4a)$$

$$\delta_2(z) = -\frac{(z - z_\alpha)^2}{2} - \frac{(z - z_\alpha)^4}{4(z_\beta - z_\alpha)^2} - \frac{2(z - z_\alpha)^3}{3(z_\beta - z_\alpha)}, \quad \delta_4(z) = -\frac{(z - z_\alpha)^4}{4(z_\beta - z_\alpha)^2} - \frac{(z - z_\alpha)^3}{3(z_\beta - z_\alpha)^2}, \quad (4b)$$

$w_\alpha(x, y, t)$ is the vertical inflow through the bottom surface of the layer and $w_\beta(x, y, t)$ is the vertical outflow through the top surface of the layer.

Eqs. (1), (3a) and (3b) are the required kinematic framework for a layer with a perhaps porous bottom and top. These equations are in terms of the horizontal velocities of the fluid. We should point out that the w velocity distribution in the water column (the z -direction) is a fourth-order polynomial and not a cubic B-spline.

3 THE CONSERVATION OF MOMENTUM

To complete the formulation of the generic model for a single layer, we need to impose the conservation of momentum, $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p / \rho + g \mathbf{e}_3 = 0$. For this we use a weak formulation. That is, we require that moments of conservation of momentum vector equation in the z -direction from z_α to z_β to be equal to zero. For this purpose, we use weighting functions $\tau_n(z) = (z - z_\alpha)^{(n-1)}$ with $n=1,4$ for the horizontal components of momentum and $n=1,5$ for the vertical components of momentum. In this process new variables are introduced, p_α , the pressure on the bottom surface, p_β , the pressure on the top surface and \bar{p}_n , $n=1,4$ the moments of the pressure through the water column from z_α to z_β . It turns out that the first moment of the vertical component of momentum is the only equation involving p_α and p_β . We require that one of these pressures be given as a boundary condition and thus the other variable becomes a dependent variable. The remaining equations (moments 1-4 in the horizontal direction and moments 2-5 in the vertical direction) form the 12 equations of motion corresponding to the 12 variables, as $u_\alpha, u_{z,\alpha}, v_\alpha, v_{z,\alpha}, u_\beta, u_{z,\beta}, v_\beta, v_{z,\beta}, \bar{p}_n, n=1,4$. If additional variables are added, such as treating z_β as the unknown instantaneous free surface, an additional kinematic boundary condition must be added for each of these variables.

The statements of the moments of the conservation of momentum equation using the kinematic framework are too complex algebraic formulations to be presented here. However, existing techniques for solving these equations have been developed over the years and have been used to compute similar GN formulations.

4 THE KINEMATIC AND DYNAMIC BOUNDARY CONDITIONS

The bottom and top surfaces of the layer are given by $\alpha(x, y, t) - z = 0$ and $\beta(x, y, t) - z = 0$, respectively. The kinematic boundary conditions for these surfaces is defined by the material derivative of these expressions or

$$\frac{\partial}{\partial t} \alpha + u_\alpha \frac{\partial}{\partial x} \alpha + v_\alpha \frac{\partial}{\partial y} \alpha - w_\alpha = 0, \quad \frac{\partial}{\partial t} \beta + u_\beta \frac{\partial}{\partial x} \beta + v_\beta \frac{\partial}{\partial y} \beta - w_\beta = 0 \quad (5)$$

We note that for a flat top or bottom surface that is a material surface, these boundary conditions become trivial.

The dynamic boundary condition for this fluid simply states that the pressure at the boundary must be resisted by, and equal to, the pressure from the other side of the boundary. For the free surface this pressure can be taken as zero. It is assumed that the bottom of the ocean is rigid and can resist any pressure from the layer in contact with it. If layers are stacked, then the pressure from the bottom of the upper layer must be transmitted to the top of the lower layer. We should note that like the requirement for the boundary pressures, p_α and p_β the leakage condition, w , can be imposed on either the top or bottom surfaces, but not both.

5 THE MGN MODEL

The 3D single-layer model consists of 12 momentum moments together with the required boundary conditions, as discussed above. Mass conservation and the boundary conditions—especially at the free surface—are satisfied exactly, while momentum is enforced in a Galerkin (weak) sense. This yields exact horizontal conservation of momentum and energy, at the expense of small errors in their vertical distribution. The formulation readily reduces to 2D and to steady (permanent-form) wave solutions. This yields exact horizontal conservation of momentum and energy, at the expense of small errors in their vertical distribution. The formulation readily reduces to 2D and to steady (permanent-form) wave solutions.

6 STACKING

As indicated in the introduction, the real power of this approach is that these layers can be stacked so regions of flow with characteristics of short length scale can be treated within a thin layer and those with larger length scales can be treated with thicker layers. This can be performed using the equations developed for a single layer. That is, no further theoretical development of the moment of momentum equations is required. Because cubic B splines are used, setting the horizontal velocities and their z -derivatives on the top surface of the bottom layer of a 2-layer configuration equal to those variables at the bottom of the top layer assures a smooth and continuous variation of all variables across the join. This stack also adds only the horizontal velocities at the top of the top layer and the integrated pressures of the top layer to the list of independent variables.

7 CONCLUSION

In this paper, we derive a Modular Green-Naghdi model, in which the horizontal velocity is expressed through B-spline shape functions with interface-based coefficients. The layerwise representation is modular and systematically refinable, allowing targeted vertical refinement near interfaces/boundaries via locally supported B-splines and enabling efficient transition from 3D to 2D, from shallow to finite to deep-water, and to stratified configurations.

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