Derivation and investigation of a scaling law for the internal wave-making resistance in a shallow two-layer fluid

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1 INTRODUCTION

When an obstacle, such as a marine vehicle or a large ice floe, moves through a stratified fluid, it can generate substantial internal waves at the pycnocline. The generation of internal waves increases the obstacle resistance. This addition is called internal wave-making resistance and is widely known as the 'dead water' effect. The earliest research on this resistance was conducted by Ekman. Subsequently, more scholars have conducted research on the factors influencing the resistance. Pite et al. [1] conducted experiments on a series of ice keel models. The results showed that the resistance peaked at $Fr \approx 0.5$, and this location was independent of the keel surface slope. Grue [2] employed a strongly nonlinear numerical model to calculate the resistance on a ship model. He found that as the ship draft increased, the resistance rised, while the *Fr* corresponding to the peak resistance decreased. The previous studies indicate that the internal wave-making resistance is highly complex, influenced by factors such as the obstacle's speed and its geometry.
Esler et al. [3] employed an asymptotic expansion method to study the drag on an orographic obstacle in a single-layer flow. Under some specific assumptions, they derived a dimensionless relationship between the drag and its influencing parameters of flow speed and obstacle height. This relationship constitutes a scaling law that effectively 'collapses' the drag for obstacles of different sizes. This law is worth extending to two-layer fluid problem. In marine engineering, this law enables rapid and accurate prediction of the internal wave-making resistance.
At last year's conference, we reanalyzed the resistance data from previously conducted towing experiments of a box model in a two-layer fluid, providing preliminary validation of the scaling law for two-layer problem. In this study, we extend the asymptotic theory from single-layer flow to the two-layer system to derive the scaling law for the internal wave-making resistance. To further validate the scaling law, we employ a strongly nonlinear numerical method developed by Grue [4]. This approach is not limited by the theory's parameter ranges, thereby permitting a comprehensive test of the scaling law's validity.

2 DERIVATION OF THE SCALING LAW

2.1 Physical scenario and governing equations

The physical scenario involves an incompressible and inviscid two-layer fluid under gravity $\tilde{g}$. The undisturbed depths are $\widetilde{H}^{\pm}$ and the constant densities are $\tilde{\rho}^{\pm}$ (+ and – superscripts denote the upper and lower layers, respectively; $\tilde{\rho}^{-} > \tilde{\rho}^{+}$). An obstacle with length $\tilde{L}$, width $\widetilde{W}$, draft $\tilde{d}$, and surface geometry $\tilde{z} = -\tilde{h}(\tilde{x}, \tilde{y})$ moves along the upper layer at a speed $\widetilde{U}$. The obstacle is symmetric about $\tilde{y}=0$. This configuration can be interpreted as a uniform two-layer flow with velocity $-\widetilde{U}$ flows past a stationary obstacle located at the upper layer boundary. A schematic of the equivalent physical scenario is provided in Figure 1.

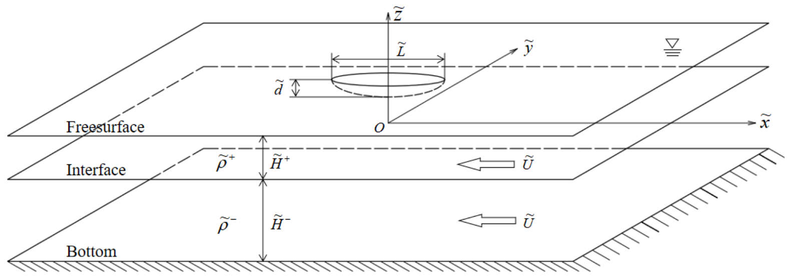

Figure 1: Schematic of the equivalent physical scenario in two-layer fluid.

The governing equations for flow in both layers are as follows:

$$\nabla \cdot \tilde{\mathbf{u}}^{\pm} = 0, \; D\tilde{\mathbf{u}}^{\pm} / D\tilde{t} = -\nabla \tilde{p}^{\pm} / \tilde{\rho}^{\pm}, \tag{1}$$

where the advective derivative $D/D\tilde{t}=\partial/\partial\tilde{t}+(\tilde{u}^{\pm}-\tilde{U})\partial/\partial\tilde{x}+\tilde{v}^{\pm}\partial/\partial\tilde{y}+\tilde{w}^{\pm}\partial/\partial\tilde{z}$; $\tilde{\mathbf{u}}^{\pm}=(\tilde{u}^{\pm}, \tilde{v}^{\pm}, \tilde{w}^{\pm})$ represent velocity components and $\tilde{p}^{\pm}$ is the hydrodynamic pressure. The flow system satisfies the following boundary conditions: (i) the kinematic and dynamic boundary conditions on the interface $\tilde{z}=\tilde{\eta}$; (ii) the no-flux boundary condition on the horizontal rigid bottom $\tilde{z}=-\tilde{H}^{-}$; (iii) the no-flux boundary condition on the obstacle surface $\tilde{z}=\tilde{H}^{+}-\tilde{h}$. Outside the region occupied by the obstacle, the free surface is assumed to be rigid (rigid-lid assumption $\tilde{w}^{+}=0$). The variables introduced above are normalized into the following dimensionless forms:

$$\tilde{u}^{\pm} - \tilde{U} = \tilde{c}_0(u^{\pm} - Fr), \; \tilde{v}^{\pm} = \tilde{c}_0 v^{\pm}, \; \tilde{w}^{\pm} = (\tilde{c}_0 \tilde{H}^{+} / \tilde{L}) w^{\pm}, \; \tilde{p}^{\pm} = \tilde{c}_0^2 \tilde{\rho}^{\pm} p^{\pm}, \; \tilde{\eta} = \tilde{H}^{+}\eta, \; \tilde{h} = \tilde{H}^{+} M h, \tag{2}$$

where $\tilde{c}_0$ is the velocity of linear long internal waves; $M=\tilde{d}/\tilde{H}^{+}$ is the dimensionless draft depth; $Fr=\tilde{U}/\tilde{c}_0$ represents the Froude number for the two-layer system. Moreover, $m=\tilde{H}^{-}/\tilde{H}^{+}$ is defined as the depth ratio of two layers and $r=\tilde{\rho}^{-}/\tilde{\rho}^{+}$ is defined as the density ratio.

2.2 Asymptotic analysis

The asymptotic regimes to be studied are: (i) the two-layer flow exhibits transcritical behavior, characterized by Fr close to 1; (ii) the obstacle possesses a large length and a small draft; (iii) the total depth of the two-layer fluid is small relative to the obstacle length, satisfying a 'shallow water' condition. Two small parameters are introduced in the present problem:

$$\varepsilon = (\tilde{d} / \tilde{H}^{+})^{2/3} = M^{2/3}, \; \mu^2 = (\tilde{H}^{+} / \tilde{L})^2. \tag{3}$$

Regimes (ii) and (iii) are characterized by the conditions $\varepsilon<<1$ and $\mu^2<<1$, respectively. The ratio between two parameters is defined as $\alpha=\mu^2/\varepsilon$. The dimensionless quantities are expanded by:

$$\mathbf{Q} = \varepsilon \mathbf{Q}_1 + \varepsilon^{2/3}\mathbf{Q}_2 + \varepsilon^2 \mathbf{Q}_3 + \ldots, \; v^{\pm} = \varepsilon^{3/2} v_1^{\pm} + \varepsilon^2 v_2^{\pm} + \varepsilon^{5/2} v_3^{\pm} + \ldots, \tag{4}$$

where $\mathbf{Q}$ denotes any one of the quantities ($u^{\pm}$, $w^{\pm}$, $p^{\pm}$, η). The two-layer fluid domain is then divided into an inner and an outer region based on the cross-stream distance from y=0. Substituting the expansions (4) into the dimensionless governing system yields a set of boundary-value problems at each order for both regions. Through the analysis of the asymptotic solutions in both regions, we ultimately derive a well-posed problem for the first-order interfacial elevation η_1^{out} in outer region as:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial \eta_1^{out}}{\partial(\varepsilon t)} - \Gamma \dfrac{\partial \eta_1^{out}}{\partial x} + \dfrac{3}{2}\dfrac{r-m^2}{m(m+r)}\eta_1^{out}\dfrac{\partial \eta_1^{out}}{\partial x} + \dfrac{1}{6}\alpha \dfrac{m+m^2 r}{m+r}\dfrac{\partial^3 \eta_1^{out}}{\partial x^3}\right) + \dfrac{1}{2}\dfrac{\partial^2 \eta_1^{out}}{\partial Y^2} = 0 \\ \dfrac{\partial \eta_1^{out}}{\partial Y}\Big|_{Y\to 0+} = Fr\displaystyle\int_0^{+\infty}\dfrac{\partial^2 h(x,\hat{y})}{\partial x^2}d\hat{y} \end{cases}, \tag{5}$$

where $\Gamma=(Fr-1)/M^{2/3}$ and $Y=\varepsilon^{1/2}y$ is a rescaled coordinate defined in outer region. According to (5), for the transcritical flow with $Fr\sim1$, η_1^{out} in steady state can be expressed functionally as:

$$\eta_1^{out} = f(\Gamma, \alpha, m, r, h(x, y)) . \quad (6)$$

2.3 The scaling law for the internal wave-making resistance

Substituting the dimensionless quantities from (2) into the formula for the internal wave-making resistance yields:

$$\tilde{F}_x = \iint_{\tilde{h}(\tilde{x},\tilde{y})\geq 0} \tilde{p}^{in+}(\tilde{x},\tilde{y}) \frac{\partial h(\tilde{x},\tilde{y})}{\partial \tilde{x}} d\tilde{x} d\tilde{y} = D(\tilde{\rho}^- - \tilde{\rho}^+)\tilde{g}(\tilde{H}^+)^2 \tilde{L} , \quad (7)$$

where D is a resistance coefficient, given by:

$$D = M \cdot \iint_{h(x,y)\geq 0} \frac{m}{m+r} p^{in+}(x,y) \frac{\partial h(x,y)}{\partial x} dxdy . \quad (8)$$

Based on the asymptotic analysis and the matching conditions between the inner and outer regions, the equivalence relation $p_1^{in+}(x)$=- $\eta_1^{in}(x)$=- $\eta_1^{out}(x,Y)|_{Y\to 0}$ can be derived. Substituting this into (8) and using the parametric expression for η_1^{out} given in (6), the parametric form of the first-order component of D can be obtained as follows:

$$D = M^{5/3} \cdot f_D(\Gamma, \alpha, m, r, h(x, y)) . \quad (9)$$

For a given surface function $h(x,y)$ and constant parameters (α,m,r), (9) reduces to the following simple form:

$$D / M^{5/3} = f_D(\Gamma) . \quad (10)$$

This is the scaling law that characterizes the dependence of the internal wave-making resistance on the obstacle speed and geometry. Its validity is governed by the parameters (α,m,r) and the surface function $h(x,y)$.

3 INVESTIGATION OF THE SCALING LAW

A strongly nonlinear numerical method originally developed by Grue [4] is employed to investigate the scaling law. This numerical method was originally implemented by Grue [2] to calculate the nonlinear resistance of a ship model. The results demonstrated its applicability to obstacles with large draft (up to $M\leq0.8$) and arbitrary horizontal scale. Consequently, this method is well-suited for validating the present scaling law, as the law itself was developed for obstacles with small draft and large length.

In order to validate the scaling law, several numerical cases are listed in Table 1. A 'case' is defined by a complete set of parameters, with the Fr varied from 0.7 to 1.3 to cover the transcritical flow regime. This variation results in a corresponding range of Γ for each case. Six such cases form a 'Group', in which the obstacle length, width and draft differ, while the parameters α, m and the surface function $h(x,y)$ remain identical. A streamlined obstacle with width-to-length ratio $\tilde{W}/\tilde{L}$=1/3 is used in this study. Here, the value of density ratio r is fixed at 1.03, which keeps consistent with the value observed in actual oceanic environments.

Table 1. Parameters of numerical cases in Group A.

Group	Case	$M=\tilde{d}/\tilde{H}^+$	$\tilde{L}/\tilde{H}^+$	$\tilde{W}/\tilde{H}^+$	α	m	$\tilde{W}/\tilde{L}$	$h(x,y)$	Range of Γ
A	1	0.2	12.09	4.03	0.02	2	1/3	$1-(2x)^2-[2y/(\tilde{W}/\tilde{L})]^2$	-0.877~ 0.877
	2	0.3	10.56	3.52					-0.669~0.669
	3	0.4	9.60	3.20					-0.553~0.553

4	0.5	8.91	2.97		-0.476~0.476
5	0.6	8.38	2.79		-0.422~0.422
6	0.7	7.96	2.65		-0.381~0.381

Figure 2(a) plots the resistance coefficient C_d ($C_d = \tilde{F}_x/(0.5\tilde{\rho}^+\tilde{S}\tilde{U}^2)$, $\tilde{S}$ the wetted area of the obstacle surface) against the Froude number *Fr* for all of the cases in Group A. The C_d-*Fr* curves for the six cases are distinct, particularly near their peak values. In contrast, Figure 2(b) presents the corresponding resistance scaled by $D/M^{5/3}$ as a function of Γ. It shows that the results for cases 1~4 merge into a unified smooth curve within the range Γ=-1.0~1.0, which encompasses nearly all nonzero resistance values. This collapse of the data effectively validates the proposed scaling law. The crest of this composite curve, marked by a dotted line, peaks at a value of 0.103. In contrast, the crests for cases 5 and 6 (dashed line) rise to 0.11, representing a 6%~7% deviation. As the parameter μ^2 in these cases (0.014 and 0.016, respectively) still satisfies $\mu^2 \ll 1$, the observed deviation at the crest is attributable to their greater drafts. This leads to the conclusion that the scaling law is most accurate for drafts of $M \leq 0.5$. Cases with deeper drafts still exhibit good data collapse elsewhere, with deviations localized to the peak region where nonlinear effects likely dominate.

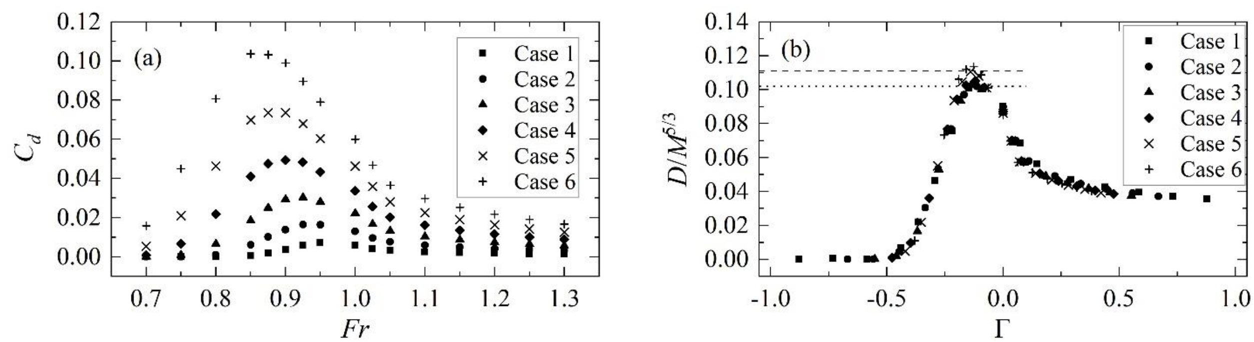

Figure 2: Resistance of cases 1~6 in Group A, scaled by (a) C_d versus *Fr*, (b) $D/M^{5/3}$ versus Γ.

4 CONCLUSIONS

In this study, a scaling law $D/M^{5/3}=f_D(\Gamma)$ is derived by extending the asymptotic theory from the single-layer flow to the two-layer fluid. This law quantitatively relates the internal wave-making resistance to its influencing factors of the obstacle speed and geometry. A strongly nonlinear numerical model was employed to validate the effectiveness of this scaling law. The results indicate that the law remains valid within the range $-1.0<\Gamma<1.0$, with optimal accuracy for draft $M \leq 0.5$. The results regarding the impact of other parameters on the validity of this scaling law will be presented at the conference.

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