Higher-order Stokes drift corrections for multichromatic, unidirectional waves in deep water

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1 Introduction

The drift associated with the motion of inviscid, irrotational water waves was derived over 150 years ago, although experimental verifications continue to pose challenges today. Called *Stokes drift,* this formally second-order quantity is derived from linear theory, and is implemented in a wide variety of wave models to calculate the motion of marine contaminants and other passive tracers.

Adhering to linear wave theory, superposition allows for the immediate generalisation of the Stokes drift from a single wave to a wave spectrum [1]. However, nonlinear effects occurring at second and third order – chief among them the appearance of bound modes – should be considered when calculating Stokes drift.

Our aim is to introduce new, analytical corrections to the Stokes drift – under assumptions of unidirectional waves and deep water for analytical simplicity – and to test these using direct numerical integration of particle paths. Velocity fields for numerical work up to third order are obtained from the reduced Hamiltonian formulation of the water-wave problem due to Zakharov, and allow for the inclusion or exclusion of bound harmonics, amplitude evolution and dispersion correction to distinguish among competing effects.

2 Mathematical background

For unidirectional, irrotational, monochromatic waves in deep water the potential correct to 2nd order in $\epsilon = ak$ can be written

$$\phi = \frac{a\omega}{k} e^{|k|z} \sin(\xi), \tag{1}$$

with $\xi = kx - \omega t$, and $\omega^2 = g|k|$. Here $z \leq 0$ and $x, t \in \mathbb{R}$.

The link between the (Eulerian) velocity fields and the (Lagrangian) particle trajectories is the system of ordinary differential equations

$$x'(t) = \phi_x(x, z, t), \tag{2}$$

$$z'(t) = \phi_z(x, z, t), \tag{3}$$

which cannot be solved analytically even for ϕ from (1), but can be numerically integrated to show that steady, periodic waves give rise to a forward drift in the direction of wave propagation. If the right-hand side of (2)–(3) is expanded in a Taylor-series about an initial position (x_0, z_0), retaining the lowest order terms gives closed, circular particle paths. If this solution is subsequently inserted into the next order, the right-hand side of (2) has a secular term, which yields the classical Stokes drift velocity

$$u_S = a^2 k\omega e^{2kz_0}. \tag{4}$$

For a wave energy spectrum $E(k) \sim a(k)^2$ this can be generalised to [1]

$$U_s = 2 \int E(k) k \omega e^{2kz} dk. \tag{5}$$

Both of these results are formally second-order, but come from *approximating linear wave theory,* which impacts their theoretical validity.

2.1 Higher-order corrections

At second order in deep water only difference harmonic terms appear in the potential [2]

$$\phi = \sum_j \frac{a_j g}{\omega_j} e^{|k_j|z} \sin(k_j x - \omega_j t) - \sum_{i>j} \omega_i a_i a_j \sin(\xi_i - \xi_j) e^{|k_i - k_j|z}. \tag{6}$$

At third order, we find both sum and difference harmonics (see [3, 4]), dispersion corrections [5] which mean ω is replaced by Ω with

$$\Omega_n = \omega_n \left[1 + \frac{\epsilon_n^2}{2} + \sum_{p<n} k_n k_p^2 a_p^2 \frac{g}{\omega_p \omega_n} + \sum_{p>n} k_n^2 k_p a_p^2 \frac{g}{\omega_p \omega_n} \right], \tag{7}$$

and a slow ($\mathcal{O}(T/\epsilon^2)$) evolution of the amplitudes, for T a typical period. On the time-scales of typical particle motions, or for stable (amplitude or energy) spectra, this time-evolution of the amplitudes can be neglected.

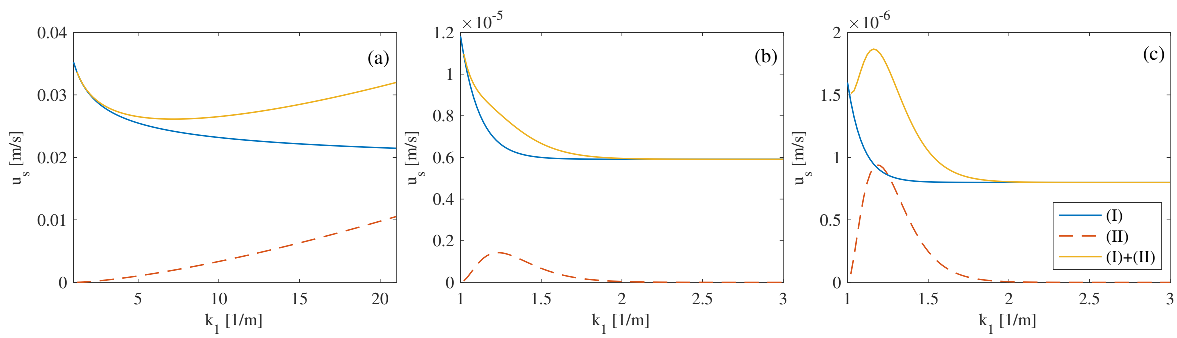

Figure 1: Illustration of the formulations (I) and (II) of (8) restricted to two modes, $k_2 = 1$ 1/m, $\epsilon_1 = \epsilon_2 = 0.075$ and variable $k_1 > k_2$, shown at a depth $z_0 = 0$ m (a), $z_0 = -4$ m (b) and $z_0 = -5$ m (c).

When the second-order potential (6) is used to calculate the Stokes drift by Taylor series expansion, following exactly the steps leading to the classical formula (4), we obtain

$$u_S = \underbrace{\sum_j a_j^2 \omega_j k_j e^{2k_j z_0}}_{\text{(I)}} + \underbrace{\sum_{k_i > k_j} \frac{\omega_i^2 a_i^2 a_j^2 (k_i - k_j)^3}{\omega_i - \omega_j} e^{2(k_i - k_j)z_0}}_{\text{(II)}}. \tag{8}$$

Term (I) in (8) is the superposition of linear Stokes drifts, while term (II) represents the contribution of difference harmonics. We note that the expression (8) can be readily extended to energy spectra along the lines of (5). Just as the Stokes drift (4) is a formally quadratic term arising from linear theory, the new term (II) is a formally quartic term arising from second order theory.

With increasing fluid depth high frequencies are preferentially damped, leading to a dominance of the higher-order (but lower frequency) difference harmonic terms for certain wavenumber combinations. Their effect on Stokes drift is evident in Figure 1, showing the constituents of (8).

3. Numerical investigation of wave-induced drift

The formulation of Stokes drift (8) can be tested using simulations of multichromatic wave trains. To this end, the particle trajectory ODEs (2)–(3) must be integrated numerically. As is the case for flume experiments [6], the drift depends on the initial position within the wave phase; overall drift must be obtained by averaging over a period, a group, or sufficiently many initial locations for mono-, bi-, or multi-chromatic waves. Suitable velocity fields with constituents up to third-order are obtained from the reduced Hamiltonian formulation of the water wave problem (calculated in [4]).

First, second, and third order theories are seen to lead to very different particle motions beneath waves, as shown for a bichromatic wave train in Figure 2. In particular, the motions at depth are significantly enhanced by the presence of bound harmonics, as shown in panel (c).

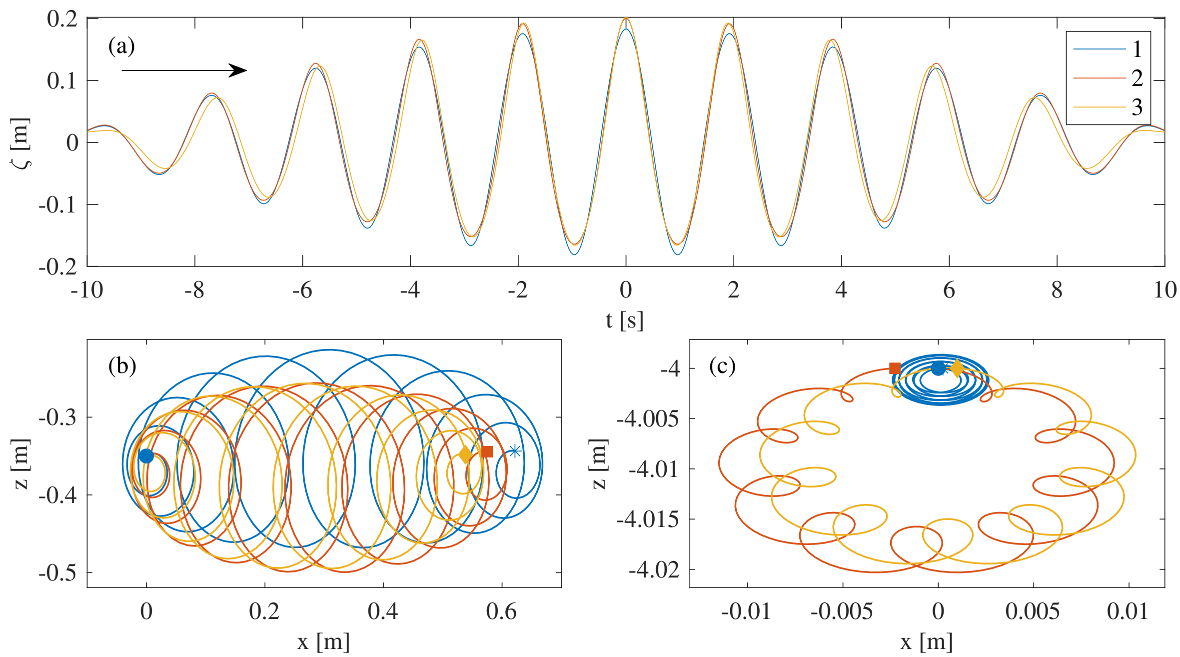

Figure 2: Time series of a bichromatic wave with $k_1 = 1.2$ and $k_2 = 1$ 1/m, $\epsilon_1 = \epsilon_2 = 0.1$ at $x_0 = 0$ (a), and accompanying particle trajectories at $z_0 = -0.35$ m (b) and $z_0 = -4$ m (c). Blue curves denote 1st order theory, red curves 2nd order theory, and yellow curves 3rd order theory in all panels.

A comparison of the Stokes drift formulation (8) and numerical Stokes drift for a bichromatic wave train is shown in Figure 3. Note that the first-order numerical result coincides almost exactly with the Stokes drift (I). Second and third-order theories give results considerably closer to our modified formulation (I)+(II).

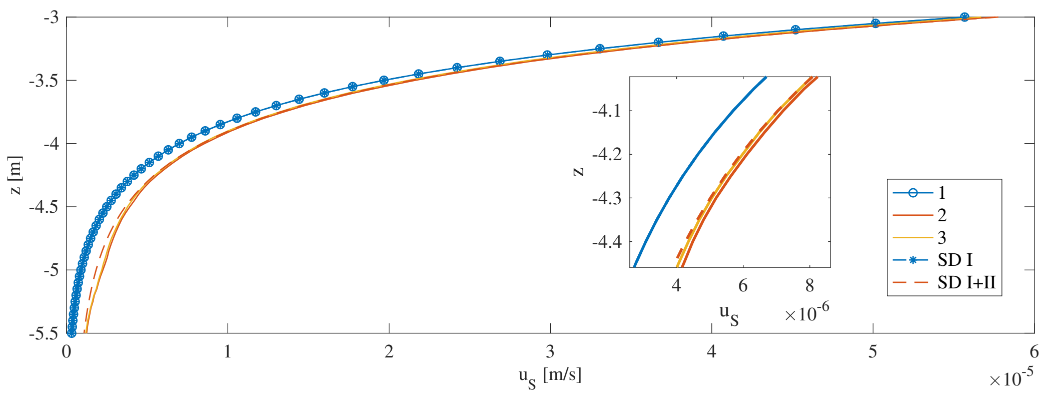

Figure 3: Comparison of Stokes drift formulations (I), (I)+(II) and drift results from numerical integration using 1st order, 2nd order, and 3rd order theory as a function of depth for a bichromatic wave train with $k_1 = 1.2$ and $k_2 = 1$ 1/m with $\epsilon_1 = \epsilon_2 = 0.075$.

3. Discussion

Higher order contributions to the Stokes drift have an effect throughout the water column. At the surface this is connected to the critical role in the Stokes drift of high frequencies [7], where dispersion corrections (7) are most influential, as well as contributions from sum-harmonic terms. At greater depths difference harmonics dominate the flow-field, and therefore the Stokes drift, as shown in work on wave groups [8]. All of this points to a need to reconsider the common formulation $u_s = a^2 k\omega \exp(2kz)$ stemming from linear wave theory.

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