Second-order hydroelastic response to localised forcing near a vertical wall

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1. Introduction

The interaction between surface waves and floating elastic plates is a classical problem in fluid mechanics with important applications in ice-covered waters, very large floating structures, and offshore engineering. In polar and coastal regions, ice sheets are frequently constrained by rigid boundaries such as harbour walls or offshore structures, leading to enhanced wave reflection and stress concentration near attachment points; linear hydroelastic models based on thin-plate theory therefore predict that bending strain typically localises near clamped boundaries [1]. However, linear theory neglects nonlinear wave interactions that become significant near hydroelastic resonance, where the forced primary wave can self-interact and generate higher harmonics. Recent studies have shown that second-order nonlinear effects may result in resonant amplification and the excitation of freely propagating second-harmonic hydroelastic waves [2, 3, 4]. In this work, we analyse weakly nonlinear hydroelastic waves generated by an oscillatory point load acting on a semi-infinite ice plate clamped to a vertical wall and show that, while linear theory predicts maximum strain near the wall, second-order effects near resonance fundamentally alter the response by shifting strain localisation towards the forcing point, where it can significantly exceed linear predictions.

2. Mathematical formulation

We consider a two-dimensional, inviscid, incompressible fluid of finite depth H' covered by a semi-infinite elastic ice plate. The plate is clamped to a vertical rigid wall at $x' = 0$ and occupies $x' > 0$. A Cartesian coordinate system (x', y') is used, with $y' = 0$ denoting the undisturbed ice–water interface and $y' = -H'$ the flat seabed.
The ice plate is modelled as a thin elastic plate with flexural rigidity D and mass per unit area M. Its vertical deflection is denoted by $y' = w'(x', t')$. The fluid motion is described by a velocity potential $\phi'(x', y', t')$ satisfying Laplace's equation

$$\nabla^2\phi' = 0, \qquad -H' \le y' \le w'(x', t'), \quad x' > 0. \tag{1}$$

At the ice–water interface, the kinematic and dynamic boundary conditions couple the fluid motion to the plate deformation. The dynamic condition combines Bernoulli's equation with the thin-plate equation. An external pressure $p'_{\text{ext}}(x', t')$ is applied to the plate, representing localised forcing.
The plate is clamped at the wall, $w'(0, t') = \frac{\partial w'}{\partial x'}(0, t') = 0$, and no-flow conditions are imposed at the wall and the seabed. The system is initially at rest.

The problem is non-dimensionalised using the characteristic hydroelastic length $L = \left(\frac{D}{\rho g}\right)^{1/4}$, where ρ is the water density.
The kinematic condition reads in the non-dimensional variables (same symbols without primes),

$$\frac{\partial w}{\partial t} = \frac{\partial \phi}{\partial y} - \varepsilon \frac{\partial \phi}{\partial x}\frac{\partial w}{\partial x} \qquad (y = \varepsilon w(x, t), \quad x > 0), \tag{2}$$

where ε is a small parameter of the problem.

The dynamic condition in the dimensionless variables takes the form (Cosserat model [5])

$$\frac{1}{\sqrt{1+\varepsilon^2 w_x^2}}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{1+\varepsilon^2 w_x^2}}\frac{\partial}{\partial x}\left(\frac{w_{xx}}{(1+\varepsilon^2 w_x^2)^{3/2}}\right)\right]+\frac{\varepsilon^2}{2}\left(\frac{w_{xx}}{(1+\varepsilon^2 w_x^2)^{3/2}}\right)^3 \tag{3}$$

$$+\alpha\frac{\partial^2 w}{\partial t^2}=-w-\frac{\partial\phi}{\partial t}-\frac{\varepsilon}{2}(\nabla\phi)^2-p_{\text{ext}}\,(x,t) \qquad (y=\varepsilon w(x,t),\quad x>0),$$

where $\alpha = M/(\rho L) = (\rho_i h_i)/(\rho L)$, ρ_i is the ice density and h_i is the ice thickness.
Introducing the small parameter $\varepsilon = w_{sc}/L = p_{sc}/(\rho g L)$ proportional to the forcing scale p_{sc} (or deflection scale w_{sc}), the governing equations are expanded using a regular perturbation approach. The plate deflection and velocity potential are written as

$$w = w^{(1)} + \varepsilon w^{(2)} + O(\varepsilon^2), \qquad \phi = \phi^{(1)} + \varepsilon\phi^{(2)} + O(\varepsilon^2). \tag{4}$$

A systematic expansion shows that geometric nonlinearities in the plate equation enter at $O(\varepsilon^2)$ and therefore do not contribute to the second-order solution. As a result, the plate can be treated as linear at this order, while nonlinear effects arise solely from the fluid–structure interaction terms.

3. Solution method

The leading-order problem describes linear hydroelastic waves forced by a time-harmonic external pressure with frequency ω. In this study, we focus on a concentrated load,

$$p_{\text{ext}}\,(x,t) = \mathrm{Re}\left\{\delta(x-\xi)\ \mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{5}$$

where ξ is the distance from the wall to the forcing location. This represents highly localised loading, such as that produced by a vehicle or machinery operating on ice.

The linear problem is solved using the vertical mode method [6] to obtain the first order deflection amplitude $W^{(1)}(x)$. The velocity potential is expanded in a series of vertical eigenfunctions that satisfy the bottom boundary condition and the coupled hydroelastic condition at the ice–water interface. The associated dispersion relation admits propagating, evanescent, and complex modes. Matching conditions are imposed at the forcing point $x = \xi$ to account for the singular load.

The second-order problem has the same linear operator as the leading-order problem but is driven by quadratic forcing terms involving products of first-order quantities. These terms generate a response consisting of:

- a double-frequency (second-harmonic) component $W^{(2)}(x)\ \mathrm{e}^{-2\mathrm{i}\omega t}$, and
- a steady (time-independent) component $W_0^{(2)}(x)$.

The second-harmonic problem is again solved using the vertical mode method, with particular solutions constructed to account for the distributed nonlinear forcing.

The second order deflection takes the form

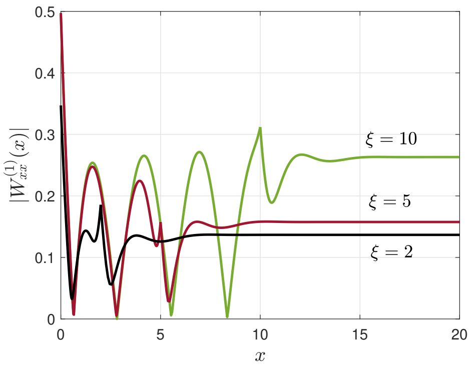
$$w(x,t) = \mathrm{Re}\left\{W^{(1)}(x)\ \mathrm{e}^{-\mathrm{i}\omega t} + \varepsilon\ W^{(2)}(x)\ \mathrm{e}^{-2\mathrm{i}\omega t} + \varepsilon\ W_0^{(2)}(x)\right\}, \tag{6}$$

where

$$W^{(1)}(x) = \begin{cases} \sum_{n=-2}^{\infty} A_n\ \mathrm{e}^{\mathrm{i}\kappa_n x} & \xi < x < \infty, \\ \sum_{n=-2}^{\infty} B_n\ \mathrm{e}^{\mathrm{i}\kappa_n x} + C_n\ \mathrm{e}^{-\mathrm{i}\kappa_n x} & 0 < x < \xi, \end{cases} \tag{7}$$

with κ_n as solutions of the dispersion relation

$$\left(\kappa^4 + 1 - \alpha\omega^2\right)\ \kappa\tanh\left(H\kappa\right) = \omega^2, \tag{8}$$



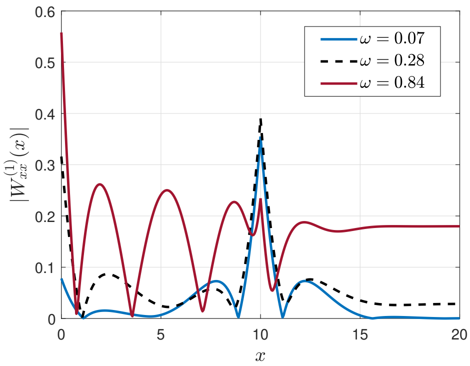


Figure 1: Magnitude of $W^{(1)}_{xx}(x)$ in the ice plate as a function of x for (a) varying values of ξ and fixed dimensionless frequency $\omega = 1.4$ and (b) varying values of ω for fixed $\xi = 10$.

and

$$\left\{W^{(2)}(x),\, W^{(2)}_0(x)\right\} = \begin{cases} \left\{W^{(2+)}_p(x),\, W^{(2+)}_{0p}(x)\right\} + \left\{W^{+}(x),\, W^{+}_0(x)\right\}, & \xi < x < \infty, \\ \left\{W^{(2-)}_p(x),\, W^{(2-)}_{0p}(x)\right\} + \left\{W^{-}(x),\, W^{-}_0(x)\right\}, & 0 < x < \xi. \end{cases} \tag{9}$$

The double-frequency component and the steady component, both are expressed as the sum of a particular solution ($W^{(2\pm)}_p(x)$, $W^{(2\pm)}_{0p}(x)$) driven by quadratic interaction terms of the first-order field and a general (homogeneous) solution ($W^{\pm}(x)$, $W^{\pm}_0(x)$) satisfying the associated dispersion relation. The particular solutions are constructed using the same vertical modal expansion as the linear problem in double series form, while the homogeneous solutions represent freely propagating and evanescent modes and are determined by enforcing continuity, jump conditions at the forcing location, and the clamped boundary conditions at the wall.

4. Results and discussion

The linear solution (Fig. 1) shows that wave reflection from the vertical wall plays a dominant role in shaping the hydroelastic response. In particular, linear theory predicts that bending strain is largest near the clamped boundary, where curvature is enhanced by the imposed kinematic constraint.

When second-order effects are included, the response changes qualitatively near hydroelastic resonance. The nonlinear self-interaction of the primary wave generates freely propagating second-harmonic waves associated with real roots of the dispersion relation. For forcing frequencies close to resonance, these second-harmonic components undergo strong amplification.

A key result is that this nonlinear amplification leads to a redistribution of strain (Fig. 2). Instead of remaining concentrated near the wall, the maximum strain shifts towards the forcing location. At resonance, the strain at the load point can significantly exceed that near the clamped edge, in stark contrast to linear predictions. This behaviour is associated with local enhancement of curvature induced by the second-order deflection. At moderate forcing frequencies, linear theory remains accurate and strain localisation near the wall persists, although second-order corrections may still modify the amplitude and spatial structure of the strain field. These findings demonstrate that classical linear models can severely underestimate local stress concentrations under resonant forcing.

5. Concluding remarks

This study examined weakly nonlinear hydroelastic waves generated by an oscillatory point load acting on a semi-infinite ice plate clamped to a vertical wall. A second-order perturbation analysis shows that geometric nonlinearities in the plate do not contribute at this order, whereas nonlinear fluid-structure interactions govern the response near hydroelastic resonance. While linear theory predicts

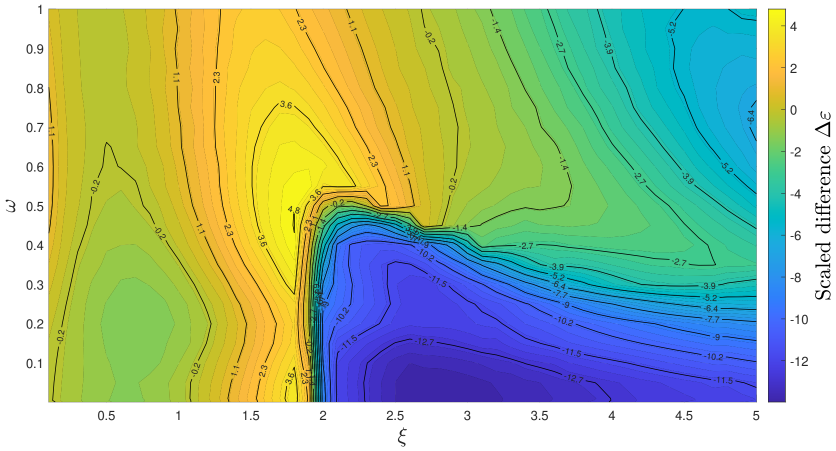


Figure 2: Contour plot showing the scaled difference $\Delta\varepsilon$ across the parameter space of ω and ξ. The quantity represents the difference between the maxima of second order strain and maxima of leading order strain normalised by yield strain ε_Y and the scale $\varepsilon_* = \frac{2L}{h_i}\frac{\epsilon_Y}{\max_{x\geq 0}|W^{(1)}_{xx}(x)|}$.

strain localisation at the clamped boundary due to wave reflection, second-order effects fundamentally alter this picture: resonant amplification of freely propagating second-harmonic waves causes the maximum strain to shift towards the forcing location, where it can substantially exceed linear estimates. The results emphasise the necessity of incorporating second-order nonlinear effects when predicting structural integrity and failure risk of ice covers and floating elastic structures subjected to dynamic, localised loading.

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