Cross-section mode method in a 3D problem of dipole oscillations in a half-frozen channel

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HIGHLIGHTS

Hydroelastic waves generated in a half-frozen channel by an oscillating submerged body located near the ice edge are investigated. The ice cover is modelled as a thin Kirchhoff-Love plate, while the submerged body is represented by a three-dimensional dipole. The solution is constructed using a mixed eigenfunction technique based on modal decompositions of the velocity potential in the channel cross-section. For the decomposition under the ice plate, the wavenumbers governing wave propagation along the channel are determined from a dispersion relation arising from a system of algebraic equations.

1 INTRODUCTION AND GOVERNING EQUATIONS

Hydroelastic interactions between surface waves and floating elastic plates are commonly analysed within linear potential-flow theory using eigenfunction expansions. Semi-infinite ice plates and related scattering problems have been studied using the vertical mode method and its extensions [1, 2, 3]. Wave generation beneath a free surface or an ice cover by submerged bodies can be modelled using oscillating singularities such as sources and dipoles [4]. We apply these approach to a half-frozen three-dimensional channel, where a dipole beneath the free surface generates waves propagating towards the ice cover.

A rectangular channel of finite depth H and half-width L is considered in Cartesian coordinates $O\tilde{x}\tilde{y}\tilde{z}$. The channel is unbounded in the $\tilde{x}$-direction (along the channel). The fluid is inviscid and incompressible, and the flow is potential. The region $\tilde{x} > 0$ is covered by a thin elastic ice plate at $\tilde{z} = 0$, while the surface is free for $\tilde{x} < 0$. The plate is clamped at $\tilde{y} = \pm L$ and has a free edge at $\tilde{x} = 0$. A submerged body in the free-surface region performs harmonic oscillations along a prescribed direction and is modelled by a dipole located at $(\tilde{x}_0, \tilde{y}_0, \tilde{z}_0)$; see Fig. 1.

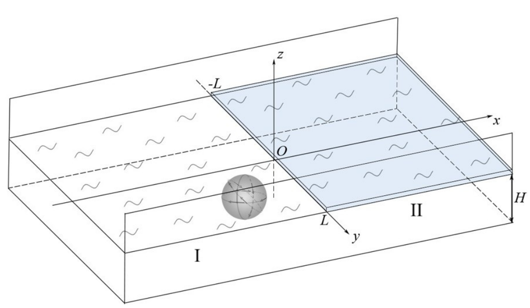

Figure 1: Schematic view of the three-dimensional problem.

Let $\tilde{\eta}(\tilde{x}, \tilde{y}, \tilde{t})$ be the free-surface elevation for $\tilde{x} < 0$ and $\tilde{w}(\tilde{x}, \tilde{y}, \tilde{t})$ the ice deflection for

$\tilde{x} > 0$. The velocity potential $\tilde{\varphi}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ satisfies Poisson's equation with dipole forcing

$$\tilde{\varphi}_{\tilde{x}\tilde{x}} + \tilde{\varphi}_{\tilde{y}\tilde{y}} + \tilde{\varphi}_{\tilde{z}\tilde{z}} = i\omega I \, \mathrm{e}^{-i\omega\tilde{t}} \, \partial_{\tilde{r}}[\delta(\tilde{x} - \tilde{x}_0)\delta(\tilde{y} - \tilde{y}_0)\delta(\tilde{z} - \tilde{z}_0)] \tag{1}$$
$$(-\infty < \tilde{x} < +\infty, -L < \tilde{y} < L, -H < \tilde{z} < 0),$$

where $-i\omega I \, \mathrm{e}^{-i\omega\tilde{t}}$ is the dipole moment and $\partial_{\tilde{r}}$ means $\partial_{\tilde{x}}$, $\partial_{\tilde{y}}$ or $\partial_{\tilde{z}}$ depending on dipole's oscillations direction. Impermeability conditions are imposed at the channel walls and bottom. At the interface between the free-surface and ice-covered regions ($\tilde{x} = 0$), matching conditions apply

$$[\tilde{\varphi}] = 0, \quad [\tilde{\varphi}_{\tilde{x}}] = 0 \qquad (\tilde{x} = 0, -L < \tilde{y} < L, -H < \tilde{z} < 0). \tag{2}$$

In addition, the ice deflection $\tilde{w}(\tilde{x}, \tilde{y}, \tilde{t})$ satisfies the thin elastic plate equation and the kinematic condition

$$M\tilde{w}_{\tilde{t}\tilde{t}} + D\tilde{\nabla}_2^4\tilde{w} = -\rho g\tilde{w} - \rho\tilde{\varphi}_{\tilde{t}}, \quad \tilde{\varphi}_{\tilde{z}} = \tilde{w}_{\tilde{t}} \qquad (\tilde{x} > 0, -L < \tilde{y} < L, \tilde{z} = 0),$$

together with the boundary conditions specified above. Here $M$is the mass of the plate per unit area, D is the plate's rigidity and ρ is the liquid's density. The free-surface elevation $\tilde{\eta}$ satisfies the dynamic and kinematic conditions

$$\rho g\tilde{\eta} + \rho\tilde{\varphi}_{\tilde{t}} = 0, \quad \tilde{\varphi}_{\tilde{z}} = \tilde{\eta}_{\tilde{t}} \qquad (\tilde{x} < 0, -L < \tilde{y} < L, \tilde{z} = 0).$$

In the regions $\tilde{x} < 0$ and $\tilde{x} > 0$, we construct the solution for the velocity potential by enforcing all boundary conditions using the cross-section mode method, which is an analogue of the vertical mode (eigenfunction) method in the 2D case.

2 METHOD OF SOLUTION

First, we introduce dimensionless variables $\tilde{x} = Lx$, $\tilde{y} = Ly$, $\tilde{z} = Lz$ and reformulate the problem in dimensionless form. The solution is sought in the form

$$\tilde{w}(\tilde{x}, \tilde{y}, \tilde{t}) = w_{sc}G(x, y)e^{-i\omega\tilde{t}}, \qquad \tilde{\eta}(\tilde{x}, \tilde{y}, \tilde{t}) = \eta_{sc}F(x, y)e^{-i\omega\tilde{t}},$$
$$\tilde{\varphi}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = -i\omega\varphi_{sc}\Phi(x, y, z)e^{-i\omega\tilde{t}}. \tag{3}$$

The scaling factors are determined from the governing equations by balancing the coefficients and prefactors. We first construct solutions of (1) in the form (3), which satisfy the kinematic and dynamic boundary conditions throughout the channel. This procedure yields the ice deflection and the corresponding strains in the ice cover.

We describe the solution for modes that are even with respect to the y-coordinate (the dipole is located on the centreline and does not oscillate in the y-direction). Odd solutions can be constructed in an analogous manner. Overall, three regions are distinguished in the channel: region I upstream of the dipole ($x < x_0$), region II between the dipole and the ice cover ($x_0 < x < 0$), and region III corresponding to the ice-covered domain ($x > 0$). In each region, the velocity potential Φ is expanded as a series of so-called cross-section modes:

In region I

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\Phi_{mn}^{fs} e^{-i\kappa_{mn}x}. \tag{4}$$

In region II

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[C_{mn}\Phi_{mn}^{fs} e^{i\kappa_{mn}(x-x_0)} + D_{mn}\Phi_{mn}^{fs} e^{-i\kappa_{mn}x}\right]. \tag{5}$$

In region III

$$\Phi = \sum_{m=1}^{\infty}\sum_{n=-2}^{\infty} B_{mn}\Phi_{mn}^{p} e^{i\xi_{mn}x}. \tag{6}$$

In the expansions (4) – (6), the first sum corresponds to the modal decomposition across the channel in the y-direction, while the second sum represents the expansion in the vertical coordinate z.

For even modes, the free-surface cross-section eigenfunctions Φ_{mn}^{fs} have the form

$$\Phi_{0n}^{fs} = \frac{1}{2}\frac{\cosh(\kappa_{0n}(z+h))}{\kappa_{0n}\sinh\kappa_{0n}h},$$

$$\Phi_{mn}^{fs} = \cos(\pi m y)\frac{\cosh(\mu_{mn}(z+h))}{\mu_{mn}\sinh\mu_{mn}h}, \quad m = 1, 2, ..., \qquad \mu_{mn} = \sqrt{\kappa_{mn}^2 + (\pi m)^2},$$

where $h = H/L$ is the dimensionless water depth, and κ_{mn} are complex-valued wavenumbers determined as roots of the channel dispersion relations

$$\kappa_{0n}\tanh(\kappa_{0n}h) = \gamma, \quad \gamma = \omega^2 L/g, \tag{7}$$

$$\mu_{mn}\tanh(\mu_{mn}h) = \gamma, \qquad m = 1, 2, \ldots. \tag{8}$$

The roots of the dispersion relation (7) are one real wavenumber κ_{00} and a countable set of imaginary wavenumbers κ_{0n}, $n \geq 1$. Roots of each dispersion relations from (8) include a countable set of imaginary wavenumbers, and one real wavenumber κ_{m0} if a periodic wave with profile $\cos(\pi m y)$ exists in the channel at the given frequency ω (no real wavenumbers otherwise). It can be showed that, as the frequency increases, the number of such waves with different profiles across the channel also increases.

To describe the velocity potential beneath the ice cover, one may (a) use an analogous cosine expansion; however, in this case the individual basis functions do not explicitly represent hydroelastic waves propagating in the frozen channel, or (b) employ functions that individually correspond to such waves. We will describe the second approach. In this case

$$\Phi_{mn}^{p}(y, z) = \sum_{q=1}^{\infty} a_q \phi_q(y, z),$$

where, in general, both the coefficients a_q and the functions ϕ_q depend on the cross-section mode indices m and n. The functions $\phi_q(y, z)$ are given by

$$\phi_q(y, z) = \frac{b_{q0}}{2}\frac{\cosh(\xi_{mn}(z+h))}{\xi_{mn}\sinh(\xi_{mn}h)} + \sum_{k=1}^{\infty} b_{qk}\cos(\pi k y)\frac{\cosh(\eta_k(z+h))}{\eta_k\sinh(\eta_k h)}, \qquad \eta_k = \sqrt{\xi_{mn}^2 + (\pi k)^2},$$

where the coefficients b_{qk} are determined from the condition $\partial_z \phi_q(y,0) = \psi_q$, where the functions ψ_q, for the ice frozen to channel walls (clamped conditions), are the normal modes of an elastic beam, satisfying $\psi_q^{(4)} = \lambda_q^4 \psi_q$, $\psi_q = \psi_q' = 0$ for $y = \pm 1$, and the set ψ_q forms an orthogonal basis.

The complex-valued wavenumbers for the plate, ξ_{mn}, are determined from a dispersion relation that follows from the algebraic system

$$\sum_{q=1}^{\infty} \Big[(1 - M_0\gamma + \beta(\lambda_q^4 + \xi_{mn}^4))\delta_{qk} - 2\beta\kappa^2 C_{qk} - \gamma M_{qk} \Big] a_q = 0, \tag{9}$$

$$C_{qk} = \int_{-1}^{1} \psi_q'' \psi_k dy, \qquad M_{qk} = \int_{-1}^{1} \phi_q(y,0)\psi_k dy,$$

where $M_0 = M/(\rho L)$ is the mass ratio and β is the dimensionless rigidity. Truncating the system (9) to M equations yields a square matrix, from which the wavenumbers ξ_{mn} can be computed for $m = 1, \ldots, M$. The resulting dispersion relation can be written as

$$f(\omega, \xi_{mn}) = \det \mathbf{A}(\omega, \xi_{mn}) = 0,$$

and the roots in the complex ξ-plane are located using the argument principle

$$\frac{1}{2\pi i} \oint_C \frac{\partial_\xi f(\omega,\xi)}{f(\omega,\xi)} d\xi = Z - P,$$

where Z and P denote the numbers of zeros and poles of f inside the contour C, respectively. Note that for modes in a channel with a free surface, the transverse modal profiles are identical for different values of the vertical index n. Similar behaviour is expected for the modes in the frozen channel. Similarly, the first mode Φ_{mn}^p with index $m = 1$ always yields one real root, corresponding to a hydroelastic wave propagating in the channel. For indices $m \geq 2$, the existence of a real root also depends on the frequency.

Integrating of nondimensional form of (1) gives boundary conditions for the potential Φ or its derivative in the plane $x = x_0$. The numbers of free-surface modes Φ^{fs} and plate modes Φ^p are chosen so as to satisfy these boundary conditions together with the interface conditions (2) and free-edge condition at $x = 0$, and to determine the expansion coefficients A_{mn}, B_{mn}, C_{mn}, and D_{mn}. Further discussions and numerical results will be presented at the workshop.

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