Wave interaction with a closely spaced array in a two-layer fluid flow problem

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Highlights

- An infinite array of closely spaced barriers over a finite-depth two-layer fluid is investigated.
- A homogenization approach is employed, wherein the small inter-barrier spacing allows the array to be modeled as an effective metamaterial.
- Two configurations are considered: non-interface-piercing and interface-piercing barriers.
- In the non-interface-piercing case, the auxiliary equation is found to have repeated roots.

1 Introduction

The interaction of water waves with infinite arrays of structures has attracted considerable research interest. Bennetts [1] investigated arrays of C-type cylinders and demonstrated broadband wave amplification, while Wilks [2] analysed graded arrays of closely spaced barriers. Huang and Porter [3] studied infinite barrier arrays using a continuum approach. Motivated by these studies, the present work examines wave interaction with an array of thin rigid barriers in a two-layer fluid.

1.1 Mathematical formulation

In this work, we consider the mathematical model of a closely spaced array of surface-piercing floating barriers in a two-layer fluid system. Linear water-wave theory and small-amplitude structural responses in finite water depth are employed to investigate the oblique interaction of waves with the breakwater configuration. Under gravity, two immiscible layers of inviscid and incompressible and fluids are assumed to undergo irrotational and simple harmonic motion.

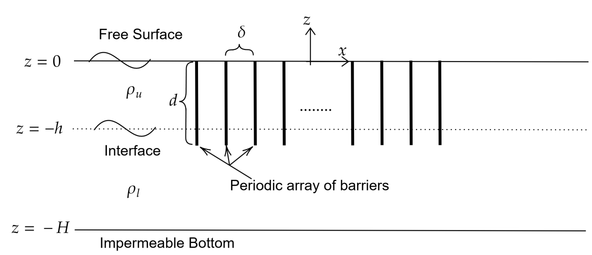

Figure 1: Array of thin rigid barriers

The interface between the two fluid layers is taken to be linear. A right-handed Cartesian coordinate system is adopted, with the z-axis pointing vertically upwards. The mean free surface of the upper fluid layer of density ρ_u is located at $z = 0$. The lower fluid layer of density $\rho_l\,(> \rho_u)$ is bounded below by a rigid bottom extending over $x \in (-\infty, \infty)$ at a mean depth $z = -H$, while the undisturbed interface between the two layers is situated at $z = -h$. Two types of barriers of uniform length d are considered at the positions $x = x_n = n\delta$ ($n \in \mathbb{Z}$). We consider an incident wave to be normally incident along the x-axis. Subsequently, the velocity potential Φ can be written as $\Phi(x, z, t) = \mathrm{Re}\left[\phi(x, z)e^{-\mathrm{i}\omega t)}\right]$ with ω as angular wave frequency. The spatial potential $\phi(x, z)$ satisfies the governing equation as follows,

$$\Delta\phi = 0, \text{ in the whole fluid domain,} \tag{1}$$

where Δ is the two-dimensional Laplacian operator. The linearised free surface condition

$$\frac{\partial \phi}{\partial z} = K\phi, \quad \text{on} \quad z = 0, \quad K = \omega^2/g, \tag{2}$$

where g is gravitational acceleration. The boundary conditions on barriers,

$$\frac{\partial \phi^{n\delta^+}}{\partial x} = \frac{\partial \phi^{n\delta^-}}{\partial x} = 0, \quad \text{on} \quad x = n\delta^{\pm}, -d < z < 0 \ \ (n \in \mathbb{Z}). \tag{3}$$

where $\delta^{\pm}$ denotes the left-right side of the barrier. The bottom boundary condition

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{on} \quad z = -H. \tag{4}$$

Matching conditions on the interface,

$$\left.\begin{aligned} \frac{\partial \phi}{\partial z}(x, -h_-) &= \frac{\partial \phi}{\partial z}(x, -h_+), \\ \rho\left(\frac{\partial}{\partial z} - K\right)\phi(x, -h_-) &= \left(\frac{\partial}{\partial z} - K\right)\phi(x, -h_+), \end{aligned}\right\} \quad \text{for} \quad x \in \mathbb{R}, \ \rho = \frac{\rho_u}{\rho_l}. \tag{5}$$

1.2 Method of solution: A continuum model

In the continuum model, we consider the gap δ to be very small, and hence the flow is irrespective to particular cell we rather consider a homogenous region. We consider there are many periodic cells are present for $x \in (0, N\delta)$. Following that our interest lies in analyzing a single cell region, i.e., $(x, z) \in (0, \delta) \times (-H, -h) \cup (-h, 0)$. We derive an approximation for wave propagation through an infinite periodic array by directly applying low frequency homogenization technique. We assume $\epsilon = \delta/d << 1$ (close spacing of barrier). In $-d < z < 0$, we make a multiple scales approximation, i.e. $x \to d\hat{x} + \delta X$. where $\hat{x}$ is the macroscale variable and X operates on the scale of a single cell. We also scale $z \to d\hat{z}$ and write

$$\phi(x, z) \sim \phi^{(0)}(\hat{x}, X, \hat{z}) + \epsilon\phi^{(1)}(\hat{x}, X, \hat{z}) + \epsilon^2\phi^{(2)}(\hat{x}, X, \hat{z}) + \dots \tag{6}$$

1.2.1 For non-interface piercing

In $0 < X < 1$ and $-1 < \hat{z} < 0$, from Eqs. (1), (2) and (3), $\phi^{(0)}$ satisfies

$$\frac{\partial^2 \phi^{(0)}}{\partial \hat{z}^2} = 0, \tag{7}$$

the solution obtained by homogenization technique can be written as $\phi^{(0)}(\hat{x}, \hat{z}) = A\, e^{\mathrm{i}\mu d\hat{x}}(1 + Kd\hat{z})$, where μ is to be determined. Outside the barrier region ($-h < z < -d$, $-H < z < -h$), the microscale is absent resulting we can rescale with $x \to d\hat{x}$ and $z \to d\hat{z}$. $\phi^{(0)}$ satisfies the laplace equation whose solution in both the regions satisfying continuity of pressure and velocity on the interface and zero-order seabed condition is $\phi^{(0)}(\hat{x}, \hat{z}) = B\, e^{\mathrm{i}\mu' d\hat{x}}\eta(\hat{z})$, where

$$\eta(\hat{z}) = \begin{cases} \frac{1}{\rho K}\Big[\cosh\mu' d(h/d + \hat{z})[\mu'(\rho - 1)\sinh\mu' d(H/d - h/d) + K\cosh\mu' d(H/d - h/d)] \\ +K\rho\sinh\mu' d(h/d + \hat{z})\sinh\mu' d(H/d - h/d)\Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -h/d < \hat{z} < -1 \\ \cosh\mu' d(\hat{z} + H/d), \qquad\qquad\qquad\qquad -H/d < \hat{z} < -h/d. \end{cases}$$

Imposing continuity of pressure and normal velocity at $\hat{z} = -1$ yields $\mu = \mu'$ and μ satisfies the equation

$$\frac{\mu[K\{\tanh(\mu p) + \rho\tanh(\mu l)\} - \mu(1-\rho)\tanh(\mu l)\tanh(\mu p)]}{K\{1+\rho\tanh(\mu p)\tanh(\mu l)\} - \mu(1-\rho)\tanh(\mu l)} = \frac{K}{1-Kd}, \tag{8}$$

where $p = h - d$ and $l = H - h$. This equation gives two real positive μ for $Kd < 1$. The corresponding mode, for non-repeated roots written in terms of the original coordinates, is

$$\phi(x,z) = \sum_{i=1}^{2} e^{i\mu_i x} \begin{cases} \dfrac{1+Kz}{1-Kd}, & -d < z < 0, \\ \dfrac{\big(\cosh(\mu_i(h+z))\big(\mu_i(\rho-1)\sinh(\mu_i l) + K\cosh(\mu_i l)\big) + K\rho\sinh(\mu_i(h+z))\sinh(\mu_i l)\big)}{\cosh(\mu_i p)\Big[(\rho-1)\mu_i\sinh(\mu_i l) + K\cosh(\mu_i l) + K\rho\sinh(\mu_i p)\sinh(\mu_i l)}, \\ & -h < z < -d, \\ \dfrac{K\rho\cosh(\mu_i(H+z))}{\cosh(\mu_i p)((\rho-1)\mu\sinh[\mu_i l) + K\cosh(\mu_i l) + K\rho\sinh(\mu_i p)\sinh(\mu_i l)}, & -H < z < -h, \end{cases}$$

1.2.2 For Interface piercing

The solution obtained by homogenization technique, in barrier region satisfying Eq. (7) and (5) is $\phi^{(0)}(\hat{x},\hat{z}) = e^{i\mu_1 d\hat{x}}\eta_1(\hat{z})$, where

$$\eta_1(\hat{z}) = \begin{cases} (1 + Kd\hat{z}) & -h/d < \hat{z} < 0, \\ (1 + (1-\rho)Kh + Kd\hat{z}) & -1 < \hat{z} < -h/d, \end{cases} \tag{9}$$

where μ_1 is a coefficient to be determined. In $-H < z < -d$ since the fluid is not bounded by any barrier, again we can drop the microscale, $\phi^{(0)}(\hat{x},\hat{z})$ satisfies the laplace equation whose solution satisfying the zero order sea bed condition is,

$$\phi^{(0)}(\hat{x},\hat{z}) = Be^{i\mu_1' d\hat{x}}\cos h\mu_1' d(H/d + \hat{z}), \tag{10}$$

applying the equation of continuity and pressure on $\hat{z} = -1$ yields $\mu_1 = \mu_1'$ and μ_1 satisfies the equation

$$\mu_1 \tanh\mu_1 d(H/d - 1) = \frac{K}{(1 + (1-\rho)Kh - Kd)}. \tag{11}$$

This equation gives one real positive μ_1 for $((Kd-\rho)Kh) < 1$ and the corresponding mode, written in terms of the original coordinates, is

$$\phi(x,z) = e^{i\mu_1 x} \begin{cases} \dfrac{1+Kz}{1+(1-\rho)Kh - Kd}, & -h < z < 0, \\ \dfrac{1+(1-\rho)Kh + Kz}{1+(1-\rho)Kh - Kd}, & -d < z < -h, \\ \dfrac{\cosh\mu_1(H+z)}{\cosh\mu_1(h-d)}, & -H < z < -d. \end{cases} \tag{12}$$

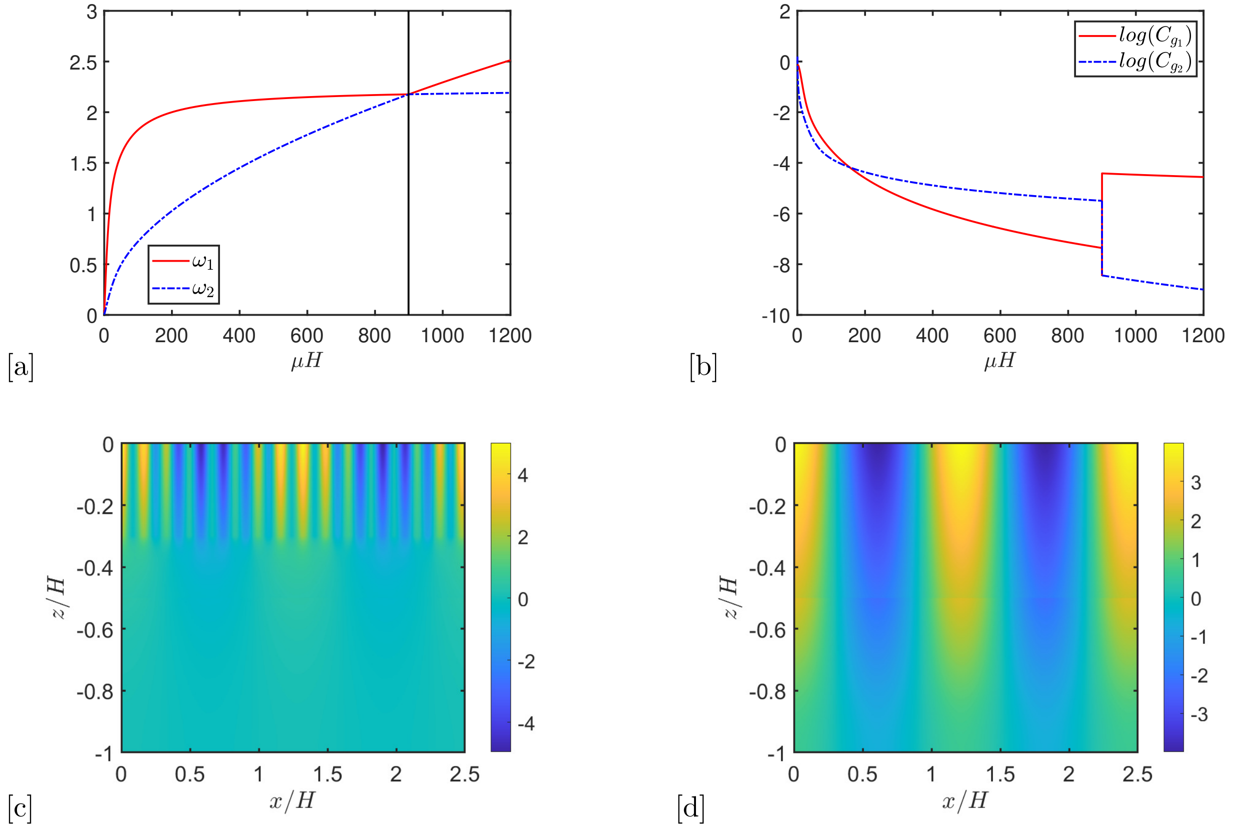

Figure 2: (a) Frequencies ω_1 and ω_2, (b) group velocities C_{g_1} and C_{g_2} with varying μH for fixed parameters $h/H = 0.5,\ d/H = 0.2,\ \rho = 0.9$, (c) contour plot of Re(ϕ) for fixed parameters, $h/H = 0.5,\ d/H = 0.3,\ \rho = 0.9, KH = 2,\ 0 < x/H < N\delta, N = 25, \delta = 0.1$ and (d) contour plot of Re(ϕ) for fixed parameters $h/H = 0.5,\ d/H = 0.8,\ \rho = 0.9, KH = 1,\ 0 < x/H < N\delta, N = 25, \delta = 0.1$.

1.3 Numerical results

In Fig. 2(a), frequencies ω_1 and ω_2 are plotted against μH, showing that for a certain value of μH, two frequencies correspond to the same μH and then diverge into different paths. This can also be understood as the roots of Eq.(8) are repeated for a frequency ω. Figure 2(b) presents the corresponding group velocities, which indicate the change of the frequencies ω_1 and ω_2 with μ. From the graph, it is observed that at the value of μH where the frequencies coincide, the corresponding group velocities undergo a rapid change. This behavior occurs because the frequency suddenly changed path, as shown in Fig. 2(a). In Fig. 2(c) and 2(d) the contour plot of Re(ϕ) is plotted, showing that Re(ϕ) exhibits a pattern of maxima and minima along the array.

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