Two alternative formulations on quadratic wave loads with forward speed effects

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1 INTRODUCTION

Accurate calculation of second-order wave loads on ships and offshore structures in the presence of current or forward speed presents at least two major challenges. The first is associated with the higher derivatives of the velocity potential that appear in the body boundary conditions when an inertial coordinate system is used. The seakeeping model formulated in a body-fixed coordinate system (BFCS) [1, 2] does not require numerical differentiation in the body boundary conditions, either in the linear or in the second-order boundary value problems (BVPs), and therefore, in principle, avoids this difficulty. However, obtaining the wave loads by integrating the quadratic pressure remains challenging because of the derivatives of the velocity potential and their quadratic combinations, particularly for offshore structures with sharp edges or for ships in oblique waves. For seakeeping models in an inertial coordinate system, alternative formulations have been developed that transform part of the body-surface integrals into surface integrals over a control surface and a portion of the free surface bounded by the body and the control surface [3]. Building on the BFCS formulation, we represent two alternative formulations for the quadratic wave loads. As a result, the most challenging part of the quadratic wave loads within the BFCS framework can be obtained without computing derivatives of the dynamic velocity potential on the body surface. The approach is applicable with the presence of forward speed and current.

2 BOUNDARY VALUE PROBLEM

For readers who are not familiar with the BFCS seakeeping formulation, a brief introduction summarized from [1] is presented. We define three coordinate systems, illustrated in Fig. 1. They are the Earth-fixed $O_eX_eY_eZ_e$, an inertial frame $OXYZ$ moving horizontally with the structure's steady velocity $\boldsymbol{W}$ in the X_eY_e-plane, and a non-inertial body-fixed frame $oxyz$ moving with both the steady and unsteady body motions. When the body is at rest, all three coordinate systems coincide with each other, with their origins located at the still-water plane and the vertical axes pointing upright.

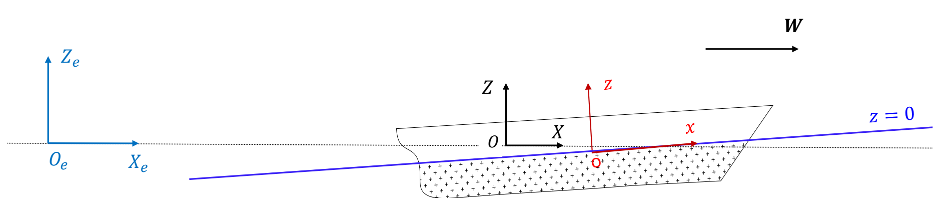

Figure 1: Definition of three coordinate systems.

The six degree-of-freedom motions of the structure in the body-fixed system $oxyz$ are defined as $\xi_i(t)$, with $i = 1, \cdots, 6$. We further define the dynamic translatory motion as $\boldsymbol{\xi}_{1-3} = (\xi_1, \xi_2, \xi_3)$ and rotational motion as $\boldsymbol{\xi}_{4-6} = (\xi_4, \xi_5, \xi_6)$. For any point $\boldsymbol{r} = (x, y, z)$ that moves with the rigid-body velocities of the structure, the induced translatory velocity is expressed as

$$\boldsymbol{u}_b = \dot{\boldsymbol{\xi}}_{1-3} + \dot{\boldsymbol{\xi}}_{4-6} \times \boldsymbol{r} = \dot{\boldsymbol{\xi}}_{1-3} + \boldsymbol{\omega} \times \boldsymbol{r}, \quad (1)$$

where an over dot means time derivative.

The linearized kinematic and dynamic free surface conditions in BFCS are

$$\left[\frac{\partial}{\partial t} - \left(\boldsymbol{W}^{(0)} - \bar{\nabla}\psi\right)\cdot\bar{\nabla}\right]\tilde{\eta} = \frac{\partial\phi}{\partial z} + (\tilde{\eta}+\bar{\eta})\frac{\partial^2\psi}{\partial z^2} - \bar{\nabla}\psi\cdot\bar{\nabla}\bar{\eta}, \qquad \text{on } z=0, \tag{2a}$$

$$\left[\frac{\partial}{\partial t} - \left(\boldsymbol{W}^{(0)} - \bar{\nabla}\psi\right)\cdot\bar{\nabla}\right]\phi = -g\tilde{\eta} + \left(\boldsymbol{\chi} + \boldsymbol{W}^{(1)}\right)\cdot\nabla\psi, \qquad \text{on } z=0. \tag{2b}$$

Here, $\tilde{\eta}(x,y,t)$ and $\phi(x,y,z,t)$ are the linear wave elevation and velocity potential respectively, including contributions from both incident and scattered waves. ψ is the local steady-flow velocity potential which could be approximated by a double-body-flow solution. $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\bar{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$ are 3D and 2D gradient operators, respectively, where $\mathbf{i},\mathbf{j}$ and $\mathbf{k}$ are the unit vector along x, y and z axis of the body-fixed coordinate system, respectively. The other variables are defined as:

$$\boldsymbol{W}^{(0)} = (U_1 - U_6 y,\ U_2 + U_6 x,\ 0), \qquad \boldsymbol{W}^{(1)} = \boldsymbol{W}^{(0)} \times \boldsymbol{\xi}_{4-6}, \tag{3a}$$

$$\boldsymbol{\chi} = \dot{\boldsymbol{\xi}}_{1-3} + \boldsymbol{\omega}\times\boldsymbol{r}, \qquad \bar{\eta} = -\left(\xi_3 - x\xi_5 + y\xi_4\right). \tag{3b}$$

Some physical explanations of these variables are provided here. $\boldsymbol{W}^{(0)}$ denotes the velocity due to steady speeds and yaw velocity, where U_1 and U_2 are the steady speed along $X-$ and $Y-$axis of the inertial coordinate system $OXYZ$ respectively, and U_6 is the steady yaw velocity around $Z-$axis. $\boldsymbol{W}^{(1)}$ is the fluctuation of the steady velocity observed in xyz frame. The latter can be understood by the fact that a constant speed in the $OXYZ$ frame becomes unsteady when observed in the $oxyz$ system which dynamically rotates with $\boldsymbol{\xi}_{4-6}$. Furthermore, $\boldsymbol{\chi}$ is the rigid-body velocity at a point $\boldsymbol{r} = (x, y, z = 0)$ on the $xy-$plane defined by $z = 0$. Different from the X_eY_e-plane of the Earth-fixed frame which does not move, the $xy-$plane moves and rotates with the dynamic motions of the structure. $\bar{\eta}$ is understood as the observed free surface movement observed in the xyz system. For instance, a positive heave motion of the structure would mean a vertically negative free surface motion in the xyz system.

The total solution is decomposed as $\phi = \phi_I + \phi_S$, where ϕ_I (known) and ϕ_S (unknown) are the velocity potentials for the incident and scattered waves, respectively. The body boundary condition is:

$$\frac{\partial\phi_S}{\partial n} = \left(\boldsymbol{u}_b + \boldsymbol{W}^{(1)} - \nabla\phi_I\right)\cdot\boldsymbol{n} \quad \text{on} \quad S_B. \tag{4}$$

Here, S_B is the body surface below $z = 0$. The velocity $\boldsymbol{u}_b$, defined already in Eq. (1), is the rigid-body velocity at a point $\boldsymbol{r}$ on S_B. $\boldsymbol{n}$ denotes the time-independent normal vector of the point on S_B in *BFCS*, pointing positive outward from the fluid domain.

3 QUADRATIC WAVE LOADS

3.1 Pressure integration

In BFCS, the total quadratic wave force (hydrostatic excluded for brevity) can be written as [1, 2]

$$\boldsymbol{F}_q = -\tfrac{1}{2}\rho\int_{S_B}\nabla\phi\cdot\nabla\phi\boldsymbol{n}\,dS + \rho\int_{S_B}\left(\boldsymbol{u}_b + \boldsymbol{W}^{(1)}\right)\cdot\nabla\phi\boldsymbol{n}\,dS + \tfrac{1}{2}\rho g\int_l(\bar{\eta}+\tilde{\eta})^2\,\boldsymbol{n}/\sqrt{1-n_3^2}\,dl. \tag{5}$$

Here l is the mean waterline defined as the intersection between S_B and the $xy-$plane. n_3 is the third component of $\boldsymbol{n}$. If the quadratic force is expressed in the inertial coordinate system XYZ, some additional terms due to coordinate transformation will appear, which however are less difficult to calculate compared with the two integrals on S_B in Eq. (5). Introducing the decomposition $\phi = \phi_I + \phi_S$, we write the total quadratic force and moment into two parts $\boldsymbol{F}_q = \boldsymbol{F}_{q,1} + \boldsymbol{F}_{q,2}$, where

$$\boldsymbol{F}_{q,1} = -\tfrac{1}{2}\rho\int_{S_B}|\nabla\phi_S|^2\,\boldsymbol{n}\,dS + \rho\int_{S_B}\left(\boldsymbol{u}_b + \boldsymbol{W}^{(1)} - \nabla\phi_I\right)\cdot\nabla\phi_S\,\boldsymbol{n}\,dS, \tag{6a}$$

$$\boldsymbol{F}_{q,2} = -\tfrac{1}{2}\rho\int_{S_B}|\nabla\phi_I|^2\,\boldsymbol{n}\,dS + \rho\int_{S_B}\left(\boldsymbol{u}_b + \boldsymbol{W}^{(1)}\right)\cdot\nabla\phi_I\,\boldsymbol{n}\,dS + \tfrac{1}{2}\rho g\int_l\frac{(\bar{\eta}+\tilde{\eta})^2}{\sqrt{1-n_3^2}}\boldsymbol{n}\,dl, \tag{6b}$$

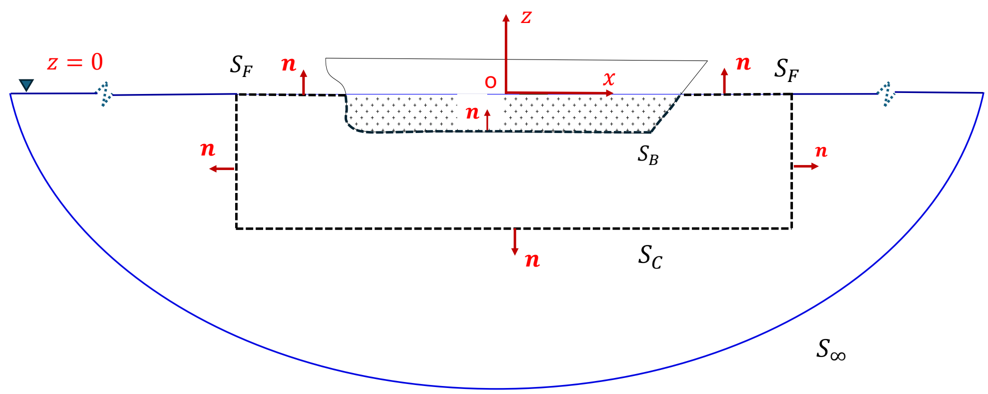

Figure 2: Definition of the fluid domain and its surrounding boundaries.

Hereafter, we will only focus on the $\boldsymbol{F}_{q,1}$ term, which involves $\nabla\phi_S$ and $|\nabla\phi_S|^2$ in its integrands, while $\boldsymbol{F}_{q,2}$ does not pose particular challenges with a standard numerical procedure.

3.2 Two new formulations

Let's consider a fluid domain $\mathcal{V}$ bounded by a closed boundary $S = S_B + S_F + S_C$, consisting: the body surface S_B, a control surface S_C enclosing S_B and a portion of the $xy-$plane between S_B and S_C. See the definition in Fig. 2. Using the following theorem from Newman [4]

$$\oint_{S_B+S_F+S_C} \left[\frac{\partial\phi_S}{\partial n}\nabla\phi_S - \tfrac{1}{2}\left|\nabla\phi_S\right|^2 \boldsymbol{n}\right] dS = 0, \tag{7}$$

and applying the body boundary condition for ϕ_S defined in Eq. (4) in Eq. (6a) lead to

$$\begin{aligned}\boldsymbol{F}_{q,1} = \rho\int_{S_B} \Big(\Big(\boldsymbol{u}_b + \boldsymbol{W}^{(1)} - \nabla\phi_I\Big)\cdot\nabla\phi_S\boldsymbol{n} - \Big(\boldsymbol{u}_b + \boldsymbol{W}^{(1)} - \nabla\phi_I\Big)\cdot\boldsymbol{n}\nabla\phi_S\Big)\, dS \\ -\rho\int_{S_F+S_C}\left(\frac{\partial\phi_S}{\partial n}\nabla\phi_S - \tfrac{1}{2}\nabla\phi_S\cdot\nabla\phi_S\boldsymbol{n}\right) dS.\end{aligned} \tag{8}$$

Again, the integral on $S_F + S_C$ does not pose any numerical challenges in practical applications, but the body surface integral does. We will present two alternative formulations that enable efficient and accurate calculation of the S_B integrals in Eq. (8), and thus for the total quadratic wave loads. Before that, we need to introduce three useful identities.

Let four vector fields $\boldsymbol{A}$, $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{B}$ be:

$$\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2, \qquad \boldsymbol{A}_1 = \dot{\boldsymbol{\xi}}_{1-3} + \boldsymbol{W}^{(1)} - \nabla\phi_I, \qquad \boldsymbol{A}_2 = \boldsymbol{\omega}\times\boldsymbol{r}, \qquad \boldsymbol{B} = \nabla\phi_S. \tag{9}$$

The integral on S_B in Eq. (8) without the fluid density for brevity, takes the following form:

$$\boldsymbol{I}_B = \int_{S_B} \left[(\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{n} - (\boldsymbol{A}\cdot\boldsymbol{n})\,\boldsymbol{B}\right] dS. \tag{10}$$

The properties of $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{B}$ in terms of divergence and curl can be summarized as:

$$\nabla\cdot\boldsymbol{A}_1 = 0, \qquad \nabla\times\boldsymbol{A}_1 = 0, \tag{11a}$$
$$\nabla\cdot\boldsymbol{A}_2 = 0, \qquad \nabla\times\boldsymbol{A}_2 = 2\boldsymbol{\omega}, \tag{11b}$$
$$\nabla\cdot\boldsymbol{B} = 0, \qquad \nabla\times\boldsymbol{B} = 0. \tag{11c}$$

Here we have used the fact that ϕ_I and ϕ_S are velocity potentials, and thus are divergence-free and curl-free. It is also obvious that $\boldsymbol{A}$ is divergence-free, but not necessarily curl-free.

With those definitions in Eq. (9) and properties in Eq. (11), the following identities can be shown:

$$\oint_S [(\boldsymbol{A}_1 \cdot \boldsymbol{B})\boldsymbol{n} - (\boldsymbol{A}_1 \cdot \boldsymbol{n})\boldsymbol{B}]\, dS = \oint_S (\boldsymbol{B} \cdot \boldsymbol{n})\, \boldsymbol{A}_1\, dS, \tag{12a}$$

$$\oint_S [(\boldsymbol{A}_2 \cdot \boldsymbol{B})\boldsymbol{n} - (\boldsymbol{A}_2 \cdot \boldsymbol{n})\boldsymbol{B}]\, dS = -\boldsymbol{\omega} \times \oint_S \phi_S \boldsymbol{n}\, dS, \tag{12b}$$

$$\oint_S [(\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{n} - (\boldsymbol{A} \cdot \boldsymbol{n})\boldsymbol{B}]\, dS = \oint_S \phi_S\, (\nabla \boldsymbol{A})\, \boldsymbol{n}\, dS. \tag{12c}$$

Here the tensor definition is implied as $[\nabla \boldsymbol{A}]_{ij} = \frac{\partial A_j}{\partial x_i}$. The mathematical derivation of these three identities is too lengthy to present in the abstract, but will be presented in the workshop.

Using $\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2$ and applying Eqs. (12a) and (12b) in Eq. (10), we get an expression for $\boldsymbol{I}_B$ and finally the first alternative formulation for $\boldsymbol{F}_{q,1}$ as

$$\begin{aligned}\boldsymbol{F}_{q,1} &= \rho \int_{S_B} \Big[\frac{\partial \phi_S}{\partial n}\, \boldsymbol{A}_1 - (\boldsymbol{\omega} \times \boldsymbol{n})\, \phi_S\Big] dS \\ &\quad + \rho \int_{S_F+S_C} \Big[\frac{\partial \phi_S}{\partial n}\, (\boldsymbol{A}_1 - \nabla\phi_S) - (\boldsymbol{A} \cdot \nabla\phi_S)\, \boldsymbol{n} + (\boldsymbol{A} \cdot \boldsymbol{n})\, \nabla\phi_S - (\boldsymbol{\omega} \times \boldsymbol{n})\, \phi_S + \tfrac{1}{2}\, |\nabla\phi_S|^2\, \boldsymbol{n}\Big] dS. \end{aligned} \tag{13}$$

Similarly, applying Eq. (12c) in Eq. (10) leads to the second alternative formulation:

$$\begin{aligned}\boldsymbol{F}_{q,1} &= \rho \oint_{S_B} \phi_S\, (\nabla \boldsymbol{A})\, \boldsymbol{n}\, dS \\ &\quad + \rho \int_{S_F+S_C} \left[\phi_S\, (\nabla \boldsymbol{A})\, \boldsymbol{n} - (\boldsymbol{A} \cdot \nabla\phi_S)\, \boldsymbol{n} + (\boldsymbol{A} \cdot \boldsymbol{n})\, \nabla\phi_S - \frac{\partial \phi_S}{\partial n} \nabla\phi_S + \frac{1}{2}\, |\nabla\phi_S|^2\, \boldsymbol{n}\right] dS \end{aligned} \tag{14}$$

Another set of identities have also been derived to develop similar formulations to the quadratic moments.

Since $\frac{\partial \phi_S}{\partial n}$ is available from the body boundary conditions, neither formulation involves integration of $\nabla\phi_S$ or $|\nabla\phi_S|^2$ on S_B. Improved accuracy and efficiency are thus expected for both formulations, which are also applicable to problems with forward speed or current. It is straightforward to adapt the above formulations for fully nonlinear potential-flow simulations, which will not be detailed here. From a numerical implementation point of view, Eq. (13) might be preferred, as Eq. (14) requires extra evaluation of terms involving $\nabla \boldsymbol{A}$, including, for instance, the second derivatives of the incident wave potential.

Both formulations have been verified using manufactured analytical solutions based on spheres in an infinite fluid domain, where a symmetric plane is considered as the S_F boundary. The first formulation has also been implemented in an in-house seakeeping code based on a time-domain high-order boundary element method [1]. More results will be reported in the workshop.

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