Nonlinear flexural-gravity waves in a channel with a porous bottom covered by an ice sheet

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HIGHLIGHTS In this work, we investigate the effect of bottom porosity on the attenuation of flexural–gravity waves in a channel with an obstruction covered by an ice sheet. The model combines potential flow theory for the fluid in the channel, the Cosserat theory of hyperelastic shells for the ice sheet, and Darcy's law for flow within the porous bottom. A fully nonlinear solution is constructed using the integral hodograph method, enabling determination of the wave attenuation rate in both subcritical and supercritical flow regimes.

1 INTRODUCTION

Flexural–gravity waves propagating beneath ice covers play an important role in the dynamics of ice-covered channels and coastal regions. These waves arise from the interaction between fluid motion, elastic deformation of the ice sheet, and variations in bottom topography. While most existing hydroelastic models assume impermeable ice and rigid or impermeable beds, natural ice covers and channel bottoms are often porous, allowing fluid penetration and inducing energy dissipation even in inviscid flows. Understanding how porosity influences wave attenuation, particularly in nonlinear regimes, is therefore essential for realistic modeling of wave-ice-bed interactions.

The effect of a porous bottom on wave propagation has been investigated in several studies, mainly within the framework of linear wave theory. Early contributions demonstrated that permeability of the seabed modifies the dispersion relation and induces wave attenuation through energy dissipation associated with Darcy-type flow within the porous medium [1]. Subsequent works examined surface gravity waves interacting with permeable rippled beds and weakly varying porous topographies, showing that bottom porosity significantly influences wave reflection, transmission, and scattering characteristics [2]. In the context of hydroelastic interactions, flexural – gravity waves over porous beds were studied using thin elastic plate models, leading to complex dispersion relations and quantification of attenuation effects due to permeability [3,4]. These studies consistently indicate that porous bottoms enhance wave-energy dissipation and alter both phase and group velocities; however, they are largely limited to linearized formulations and do not address finite-amplitude nonlinear effects, channel confinement, or the combined influence of porous ice covers and porous beds.

Fully nonlinear hydroelastic wave problems in channels remain relatively unexplored, especially in the presence of dissipation induced by porosity. In particular, the combined effects of ice porosity, bottom porosity, and finite-amplitude nonlinearity on flexural–gravity waves in channel flows have not yet been investigated.

In this work, we investigate fully nonlinear flexural–gravity waves in a channel with a porous bottom covered by a porous ice sheet. The flow is modeled as inviscid, incompressible, and irrotational, while the ice cover is described using the Cosserat theory of hyperelastic shells. Porosity of both the ice and the bottom is incorporated through Darcy's law, providing a physically consistent mechanism for wave attenuation. A fully nonlinear solution is developed using the integral hodograph method, enabling determination of wave profiles and attenuation rates in both subcritical and supercritical regimes. The results demonstrate how nonlinear effects and channel confinement interact with porous dissipation, yielding attenuation behavior that differs fundamentally from linear predictions.

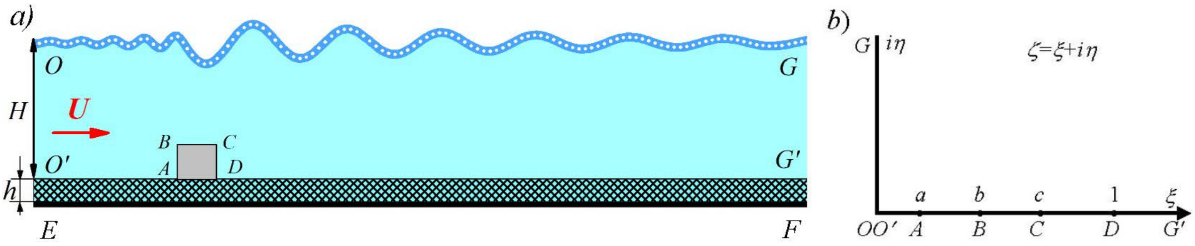

Figure 1. (a) Schematic of flow in a channel *OGG'O'* with an obstruction *ABCD*, a porous bed *O'G'FE*, and a porous ice sheet of thickness *δ*, (b) parameter, or ζ- plane.

2 FORMULATION OF THE PROBLEM

We consider a two-dimensional steady flow in a channel *OGG'O'* containing a fixed obstruction *ABCD* resting on a porous bottom *O'G'FE*. The channel is covered by an elastic porous plate modeling an ice sheet. A Cartesian coordinate system *XY* is introduced with the origin located at the center of the obstruction. Far upstream and downstream, the flow is assumed to be uniform with velocity *U*. A schematic of the flow configuration and coordinate system is shown in Figure 1a. The fluid in the channel is inviscid and incompressible, and the flow is assumed to be irrotational, allowing the use of a potential flow formulation. Porosity of both the bottom and the ice sheet is incorporated through Darcy's law, which relates the fluid penetration velocity into the porous media to the local pressure at the bed–liquid and ice–liquid interfaces. The obstruction is characterized by a horizontal length a and height b, while the thickness of the porous bottom is denoted by h.

Complex potential. We introduce the complex velocity potential $w(z) = \phi(x,y) + i\psi(x,y)$, where $\phi(x,y)$ is the velocity potential, $\psi(x,y)$ is the stream function, and $z = x + iy$ is the complex spatial variable. The objective is to determine the function $w[z(\zeta)]$ which which conformally maps a parameter plane ζ onto the complex-velocity potential domain. We choose the first quadrant of the ζ −plane shown in figure 1b, as the parameter region and derive expressions for the nondimensional complex velocity, dw/dz, and for the derivative of the complex potential $dw/d\zeta$, both as functions of the variable ζ. Once these functions are obtained, the velocity field and the mapping between the parameter plane and the physical flow domain can be determined as follows:

$$z(\zeta) = z_0 + \int_0^{\xi} \frac{dw}{d\zeta'} \Big/ \frac{dw}{dz} d\zeta'. \tag{1}$$

We apply the integral hodograph method to determine expressions for the complex velocity and the derivative of the complex potential. The boundary conditions for the complex velocity are prescribed in terms of the velocity direction $\beta(\xi)$along the channel bottom, corresponding to the real axis of the parameter plane, and the velocity magnitude $v(\eta)$along the ice–liquid interface, corresponding to the imaginary axis of the parameter plane. Using the integral formula for solving this mixed boundary-value problem [5], the complex velocity is obtained as

$$\frac{dw}{dz} = v_0 \sqrt{\frac{\zeta-a}{\zeta+a}\frac{\zeta+b}{\zeta-b}\frac{\zeta+c}{\zeta-c}\frac{\zeta-1}{\zeta+1}} \exp\left[\frac{1}{\pi}\int_{0,1}^{a,\infty} \frac{d\beta}{d\xi} \ln\left(\frac{\xi-\zeta}{\xi+\zeta}\right) d\xi - \frac{i}{\pi}\int_0^{\infty} \frac{d\ln v}{d\eta} \ln\left(\frac{i\eta-\zeta}{i\eta+\zeta}\right) d\eta\right], \tag{2}$$

where $v_0$denotes the velocity magnitude at point O, $\beta(\xi) = \arctan(v_y/v_x)$, and v_x and v_y are the horizontal and vertical components of the velocity along the channel bottom, respectively. The vertical velocity component will be determined later by accounting for flow within the porous bed. The parameters a, b, and c correspond to the points A, B, and C in the physical plane.

To obtain an expression for the derivative of the complex potential in the parameter plane, $dw/d\zeta$, it is convenient to introduce the unit vectors $\mathbf{n}$ and $\mathbf{t}$ on the fluid boundary, which denote the outward normal and tangential directions to the surface, respectively. With this notation, we have

$$dw = (v_\tau + i v_n) ds, \tag{3}$$

where $v_\tau$and $v_n$are the tangential and normal components of the velocity on the surface, respectively. Let

$$\theta(\eta) = \tan^{-1}\left(\frac{v_n}{v_\tau}\right) \quad \text{and} \quad \gamma(\xi) = \tan^{-1}\left(\frac{v_{bn}}{v_{b\tau}}\right)$$

denote the angles between the velocity vector and the tangential direction on the ice–liquid interface and the channel bottom, respectively. Equation (3) then allows us to determine the argument of the derivative of the complex potential, $\vartheta = \arg(dw/d\zeta)$

$$\vartheta(\zeta) = \arg\left(\frac{dw}{d\zeta}\right) = \arg\left(\frac{dw}{ds}\right) + \arg\left(\frac{ds}{d\zeta}\right) = \begin{cases} \gamma(\xi), & 0 < \xi < \infty, \quad \eta = 0, \\ \theta(\eta) + \pi/2, & \xi = 0, \quad 0 < \eta < \infty. \end{cases} \tag{4}$$

By determining the angles $\gamma(\xi)$and $\theta(\eta)$along the channel bottom and the ice–liquid interface, respectively, and applying the integral formula for solving a uniform boundary-value problem for a complex function, we obtain the following expression for the derivative of the complex potential in the ζ-plane:

$$\frac{dw}{d\zeta} = \frac{2}{\pi\zeta} \exp\left[\frac{1}{\pi}\int_0^\infty \frac{d\gamma}{d\xi} \ln(\xi^2 - \zeta^2)\, d\xi + \frac{1}{\pi}\int_0^\infty \frac{d\theta}{d\eta} \ln(\eta^2 + \zeta^2)\ d\eta\right]. \tag{5}$$

Dividing (5) by (2), we can obtain the derivative of the mapping function

$$\frac{dz}{d\zeta} = \frac{dw}{d\zeta} \Big/ \frac{dw}{dz} = \frac{2}{\pi\zeta} \exp\begin{bmatrix} \frac{1}{\pi}\int_0^\infty \frac{d\gamma}{d\xi} \ln(\xi^2 - \zeta^2)\, d\xi + \frac{1}{\pi}\int_0^\infty \frac{d\theta}{d\eta} \ln(\zeta^2 + \eta^2)\, d\eta \\ -\frac{1}{\pi}\int_0^\infty \frac{d\beta}{d\xi} \ln\left(\frac{\xi - \zeta}{\xi + \zeta}\right) d\xi + \frac{i}{\pi}\int_0^\infty \frac{d\ln v}{d\eta} \ln\left(\frac{i\eta - \varsigma}{i\eta + \varsigma}\right) d\eta \end{bmatrix}, \tag{6}$$

The functions $v(\eta)$, $\gamma(\xi)$, $\theta(\eta)$, and $\beta(\xi)$ are determined from the dynamic boundary condition at the ice/liquid interface, the permeability conditions of both the bottom and the ice sheet, and the prescribed geometry of the channel bottom.

Porous model. We assume slow motion of the fluid within the porous bottom and the ice sheet, which is governed by Darcy's law.

$$\boldsymbol{v} = -\frac{K}{\mu}\, grad\,(p - \rho g y)\,, \tag{7}$$

where $K$is the permeability coefficient, μ is the dynamic viscosity, ρ is the fluid density, g is the acceleration due to gravity, and p denotes the pressure. For a thin ice sheet, $h_i \ll 1$, equation (7) can be integrated across the ice sheet, yielding the normal component of the velocity as

$$v_n^{ice} \;=\; -\delta_{ice} p(x), \tag{8}$$

where $\delta_{\text{ice}} = K/(\mu h_i)$, and the hydrostatic term $-\rho g y$ is incorporated into $p(x)$, which represents the pressure at the ice/liquid interface. At the channel bottom beneath the porous layer, the pressure is assumed to be constant and equal to p_b. The normal component of the velocity is then given by

$$v_n^{bot} \;=\; -\delta_{bot}[p(x) - p_b], \qquad \delta_{bot} = K/(\mu b). \tag{9}$$

System of equations

The governing equations (2), (5), and (6) involve the parameters a, b, and c, as well as the unknown functions $v(\eta)$, $\gamma(\xi)$, $\theta(\eta)$, and $\beta(\xi)$. The parameters a, b, and c are determined from the linear dimensions of the obstruction.

$$\int_{a,b,c}^{1} \frac{ds}{d\xi} d\xi = S_{\{AD,BD,CD\}}, \tag{10}$$

where $\frac{ds}{d\xi} = \left|\frac{dz}{d\zeta}\right|_{\zeta=\xi}$, and $S_{\{AD,BD,CD\}}$ are the arclengths from point D to points A, B, C.

Taking the argument of $(dz/d\zeta)$, we obtain the relation between the angles γ, β on the bottom,

$$\arg\left(\frac{dz}{d\xi}\right)_{\zeta=\xi} = \arg\left(\frac{dw}{d\xi}\right)_{\zeta=\xi} - \arg\left(\frac{dw}{dz}\right)_{\zeta=\xi} = \gamma(\xi) + \beta(\xi) = 0, 0< \xi < a, \quad 1< \xi < \infty. \tag{11}$$

The tangential component of the velocity v_τ on the bottom is obtained as follows

$$v_\tau^{bot}(\xi) = Re\left(\frac{dw}{dz}\frac{dz}{ds}\right) = Re\left(\left.\frac{dw}{dz}\right|_{\zeta=\xi}\right), \tag{12}$$

Then, using the definition of the angle $\gamma(\xi)$, we obtain

$$\gamma(\xi) = \arctan\frac{v_n}{v_\tau} = \arctan\left(\frac{1}{Re(dw/dz|_{\zeta=\xi})}\right). \tag{13}$$

The function $\theta(\eta)$ along the ice/liquid interface is obtained as

$$\theta(\eta) = \arcsin\frac{v_n^{ice}(\eta)}{v(\eta)}. \tag{14}$$

The system of equations (10) - (14) allows us to determine the parameters a, b, c and the functions $\gamma(\xi)$, $\beta(\xi)$, and $\theta(\eta)$. The function $v(\eta)$ is determined numerically. In discrete form, the solution is sought on two fixed sets of points: a set $-\xi^* < \xi_j < \xi^*, j = 1,\dots,N$ corresponding to the bottom of the channel and a set $-\eta^* < \eta_i < \eta^*$, $i = 1,\dots,M$ corresponding to the interface; both sets of points ξ_j and η_i monotonically increase. By applying the dynamic boundary condition at the points $\eta_k, k = 1,\dots,\bar{K}$, we can obtain the following system of nonlinear equations

$$G_k(\bar{V}) = c_{pk}(\bar{V}) - c_{pk}^{ice}(\bar{V}) = 0, \qquad k = 1,\dots,\bar{K}, \tag{15}$$

where $\bar{V} = (v_1, v_2, \dots, v_{\bar{K}})^T$ is the vector of the unknown velocities v_k;

$$c_{pk}(\bar{V}) = 1 - v_k^2 - \frac{2[y_k(\bar{V})-1]}{F^2}; \qquad c_{pk}^{ice}(\bar{V}) = 2D\left[\left(\frac{d^2\kappa}{ds^2}\right)_k + \frac{1}{2}\kappa_k^3\right]. \qquad (16)-(17)$$

are the hydrodynamic pressure coefficient and the pressure coefficient due to the elastic sheet, respectively. The system of equations (15) is solved using Newton's method.

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