Scattering of internal pressure waves by an underwater step in a compressible ocean

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HIGHLIGHTS

- Model internal pressure wave scattering by a step in a weakly compressible fluid.
- A solution is found using eigenfunction matching with a custom inner product.
- Observe an intricate reflection pattern developed from an initial Gaussian pressure.

1 INTRODUCTION

Underwater explosions and volcanic eruptions generate pressure waves that travel through the ocean interacting with the seabed and the ocean surface. Such pressure waves have been used to estimate the crash site of Malaysian Airlines Flight 370 (MH370)[1]. The inclusion of compressibility is necessary to accurately model internal pressure waves, as they give rise to acoustic-gravity waves [2].

We use an eigenfunction matching approach to model the propagation of acoustic gravity waves generated by an initial pressure distribution [3] (a stand in for the more complicated short-time behaviour of an explosion) near a sudden depth change in the bathymetry [4].

2 MATHEMATICAL MODEL AND SOLUTION

We consider a weakly compressible, inviscid, and irrotational fluid between a free-surface and an impermeable bottom with a step-like configuration. The fluid is given some initial pressure distribution $P_0(x, z)$ that can be thought of as the result of an underwater explosion. The motion of the fluid can be represented by its velocity potential $\Phi(x, z, t)$ within the fluid domain Ω (Figure 1). The governing equations are given by

$$\Phi_{tt} = c^2 \nabla^2 \Phi, \qquad\qquad (x, z) \in \Omega, \qquad\qquad (1a)$$

$$\Phi_{tt} + g\Phi_z = 0, \qquad\qquad z = 0, \qquad\qquad (1b)$$

$$\Phi_n = 0, \qquad\qquad (x, z) \in \partial\Omega_b, \qquad\qquad (1c)$$

$$\Phi_x \to 0, \qquad\qquad |x| \to \infty, \qquad\qquad (1d)$$

subject to the initial conditions

$$\Phi(x, z, 0) = \Phi_0(x, z), \qquad \Phi_t(x, z, 0) = -\frac{P_0(x, z)}{\rho}, \qquad\qquad (1e)$$

where the subscripts of Φ indicate partial differentiation, Φ_n is the normal derivative of Φ with respect to the bottom $\partial\Omega_b$, c is the speed of sound in the fluid, g is acceleration due to gravity, and ρ is the density of the fluid.
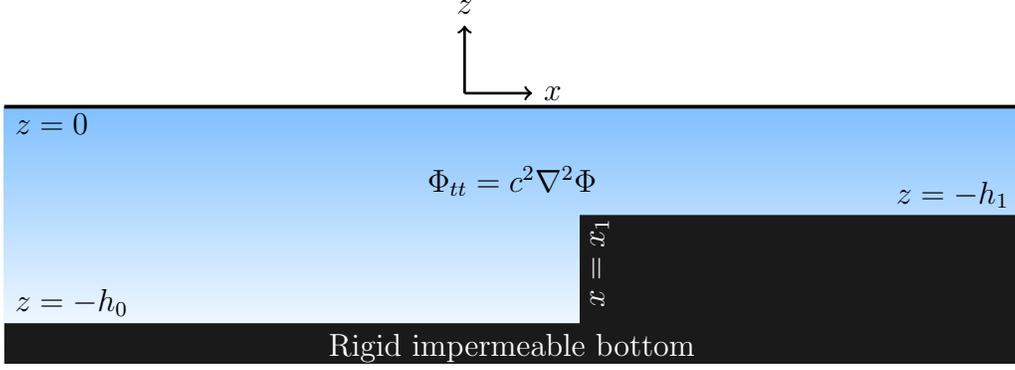
Figure 1: Schematic diagram of the physical problem showing wave propagation in a compressible ocean due to an underwater explosion.

The governing equations (1) are in the form of a generalise wave equation with an unbounded x-domain and a restricted z-domain, the solution will be of the form

$$
\Phi(x,z,t) = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{\langle \Phi_0(x,z), \phi_n^- \rangle_{\mathcal{E}}}{D_n^-} \phi_n^- \cos(\omega_n^- t) + \frac{\langle \Phi_0(x,z), \phi_n^+ \rangle_{\mathcal{E}}}{D_n^+} \phi_n^+ \cos(\omega_n^+ t)\, \mathrm{d}k
$$
$$
- \frac{1}{\rho} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{\langle P_0(x,z), \phi_n^- \rangle_{\mathcal{E}} \phi_n^-}{D_n^-} \frac{\sin(\omega_n^- t)}{\omega_n^-} + \frac{\langle P_0(x,z), \phi_n^+ \rangle_{\mathcal{E}} \phi_n^+}{D_n^+} \frac{\sin(\omega_n^+ t)}{\omega_n^+}\, \mathrm{d}k, \quad (2)
$$

where $\phi_n^{\pm}(x,z;k)$ are the eigenfunctions that represent the incident waves moving on the right $(+)$ and left $(-)$ for the vertical mode n-th and horizontal wavenumber k, $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is an inner product that diagonalises the governing equations, $D_n^{\pm} = \langle \phi_n^{\pm}, \phi_n^{\pm} \rangle_{\mathcal{E}}$ is the normalisation coefficient for the eigenfunctions, and $\omega_n^{\pm}(k)$ is the wave frequency.

The eigenfunctions are found by separating the domain into regions of constant depth and using separation of variables. The resulting eigenfunctions are

$$
\phi_n^+(x,z;k) = \begin{cases} e^{-\mathrm{i}k(x-x_1)} \dfrac{\cosh \mu_n^+(z+h_0)}{\cosh(\mu_n^+ h_0)} + \displaystyle\sum_{i=0}^{\infty} C_{i,n}^+ e^{\mathrm{i}k_{i,n}^{+(0)}(x-x_1)} \dfrac{\cosh \mu_{i,n}^{+(0)}(z+h_0)}{\cosh(\mu_{i,n}^{+(0)} h_0)} & x < x_1 \\[3ex] \displaystyle\sum_{i=0}^{\infty} B_{i,n}^+ e^{-\mathrm{i}k_{i,n}^{+(1)}(x-x_1)} \dfrac{\cosh \mu_{i,n}^{+(1)}(z+h_1)}{\cosh(\mu_{i,n}^{+(1)} h_1)} & x > x_1 \end{cases},
$$
$$(3)$$

$$
\phi_n^-(x,z;k) = \begin{cases} \displaystyle\sum_{i=0}^{\infty} C_{i,n}^- e^{\mathrm{i}k_{i,n}^{-(0)}(x-x_1)} \dfrac{\cosh \mu_{i,n}^{-(0)}(z+h_0)}{\cosh(\mu_{i,n}^{-(0)} h_0)} & x < x_1 \\[3ex] e^{\mathrm{i}k(x-x_1)} \dfrac{\cosh \mu_n^-(z+h_1)}{\cosh(\mu_n^- h_1)} + \displaystyle\sum_{i=0}^{\infty} B_{i,n}^- e^{-\mathrm{i}k_{i,n}^{-(1)}(x-x_1)} \dfrac{\cosh \mu_{i,n}^{-(1)}(z+h_1)}{\cosh(\mu_{i,n}^{-(1)} h_1)} & x > x_1 \end{cases},
$$
$$(4)$$

where μ_n is found by satisfying the dispersion relation

$$
\mu_n^{\pm} \tanh \mu_n^{\pm} h^{\pm} + \frac{c^2}{g^2} \mu_n^{\pm 2} = \frac{c^2 k^2}{g^2}, \tag{5}
$$

where $h^+ = h_0$ and $h^- = h_1$ are the far-field depths of the incoming wave, and $\mu_{i,n}^{\pm(j)}$ and $k_{i,n}^{\pm(j)}$ are given by

$$\frac{\omega_n^{\pm 2}}{g} = \mu_{i,n}^{\pm(j)} \tanh \mu_{i,n}^{\pm(j)} h_j \quad \text{and} \quad k_{i,n}^{\pm(j)2} = \frac{\omega_n^{\pm 2}}{c^2} + \mu_{i,n}^{\pm(j)2}, \quad \text{with} \quad \omega_n^{\pm 2} = c^2(k^2 - \mu_n^{\pm 2}), \quad (6)$$

where $k_{i,n}^{\pm(j)}$ is either a positive real number or an imaginary number with a negative imaginary component.

The coefficients $B_{i,n}^{\pm}$ and $C_{i,n}^{\pm}$ are found by applying the matching conditions

$$\lim_{x \to x_1^-} \phi_n^{\pm}(x, z; k) = \lim_{x \to x_1^+} \phi_n^{\pm}(x, z; k) \quad \text{and} \quad \lim_{x \to x_1^-} \frac{\partial \phi_n^{\pm}}{\partial x} = \lim_{x \to x_1^+} \frac{\partial \phi_n^{\pm}}{\partial x},$$

taking the standard L^2 inner product of the potential matching in the z-direction for $z \in (-h_1, 0)$ with respect to $\cosh\left(\mu_{i,n}^{+(1)}(z + h_1)\right) / \cosh\left(\mu_{i,n}^{+(1)} h_1\right)$, and the inner product of the velocity matching for $z \in (-h_1, 0)$ with respect to $\cosh\left(\mu_{i,n}^{+(0)}(z + h_0)\right) / \cosh\left(\mu_{i,n}^{+(0)} h_0\right)$ to generate a system of linear equations that can be solved numerically.

Finally, following [3] we define the general inner product

$$\langle \Phi, \Psi \rangle_{\mathcal{E}} = \int_{\Omega} \Phi(x, z) \overline{\Psi(x, z)} \, \mathrm{d}x\mathrm{d}z + \frac{c^2}{g} \int_{-\infty}^{\infty} \Phi(x, 0) \overline{\Psi(x, 0)} \, \mathrm{d}x, \quad (7)$$

from which we can perform the inner products in (2) and noting that the normalisation coefficient is given by

$$D_n^{\pm} = 2\pi \left(\frac{h^{\pm}}{2} \operatorname{sech}^2 \mu_n^{\pm} h^{\pm} + \frac{\tanh \mu_n h^{\pm}}{2\mu_n^{\pm}} + \frac{c^2}{g} \right).$$

3 RESULTS

We present solutions for the internal pressure waves and the free-surface elevation given by

$$P(x, z, t) = -\rho \Phi_t(x, z, t), \quad \text{and} \quad \eta(x, t) = \frac{1}{\rho g} P(x, 0, t), \quad (8)$$

respectively, created by an initial Gaussian pressure distribution;

$$P_0(x, z) = \hat{P} e^{-\pi^2 \frac{(x - x_c)^2 + (z - z_c)^2}{\sigma^2}}, \quad (9)$$

and $\Phi_0(x, z, 0) = 0$, near a step change in bathymetry as shown in Figure 1. To evaluate (2) numerically we must truncate the infinite sum and integral, we chose to truncate the sum after $N = 60$ modes and wave number $k_{\max} = 0.2$. We then uniformly discretise the integral with 8001 points and evaluate the integral using trapezoidal rule quadrature. We choose $\hat{P} = 10^6$ Pa, $(x_c, z_c) = (0, -3000)$ m, $g = 9.81 \text{m/s}^2$, $c = 1450 \text{m/s}$, $x_1 = 2000 \text{m}$, $h_0 = 4000 \text{m}$, $h_1 = 2000 \text{m}$ and $\sigma = 200 \text{m}$.

Figure 2 shows how the initial pressure disturbance develops after $t = 2.5$s and 7.5s. We can see that the pressure wave has expanded radially away from the initial location and reflected off the bottom. The wave also reflects off of the free-surface while undergoing a phase change. The surface response to the reflected pressure waves, while initially simple, at later times presents an intricate pattern highly dependent on the underlying bathymetry.
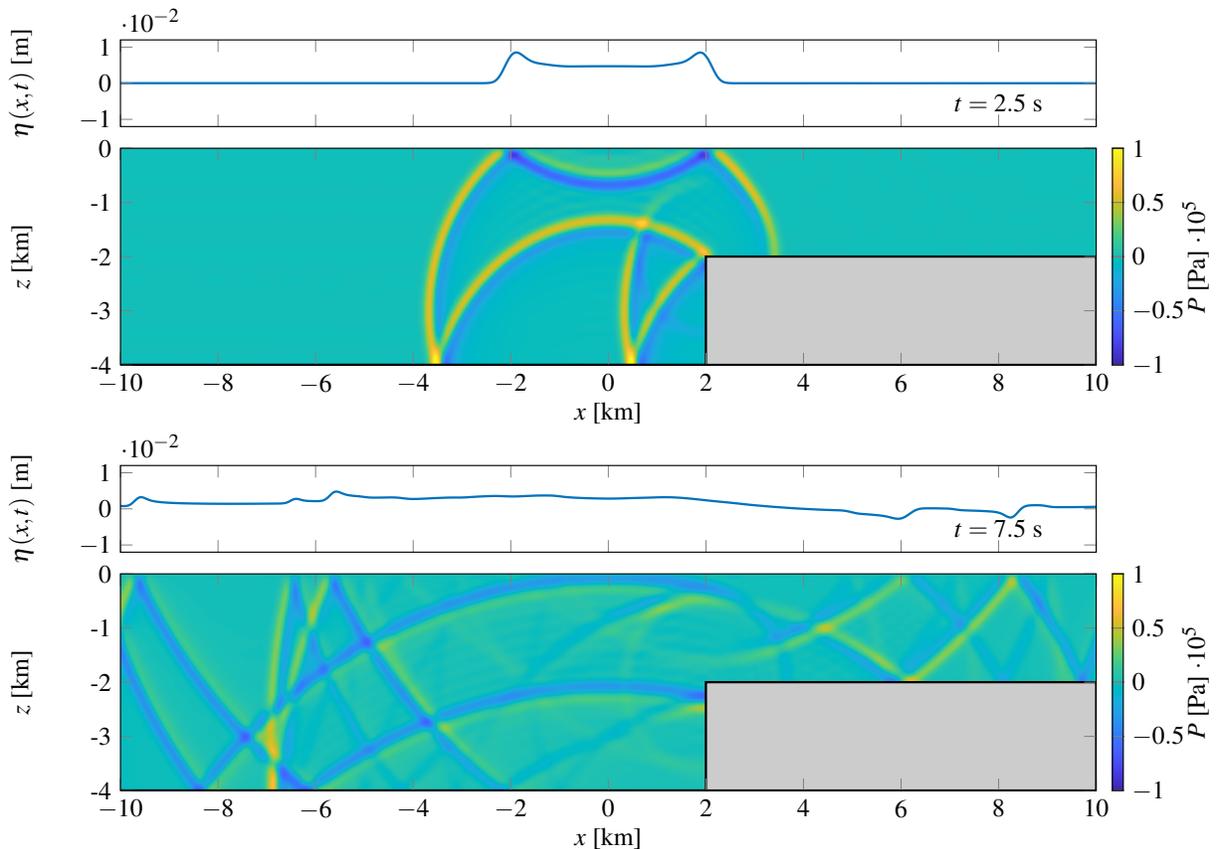
Figure 2: Compression waves caused by an initial Gaussian pressure distribution centred at $(0, -3000)$.

4 CONCLUSION

We have investigated the propagation and scattering of compressible pressure waves generated by an initial Gaussian distribution. We have observed the complicated reflection patterns caused by a step bathymetry. Due to the dependence on bathymetry and the initial location of the pressure, more research is required to fully understand the effects of this scattering. Generalisation to an arbitrary piecewise-constant bathymetry can be achieved with careful consideration of the existence of bound wave modes.

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