

Rankine-Fourier decomposition of the NN integral equation

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Highlight: Application of the fundamental Rankine-Fourier decomposition of the Green function in the basic NN (Neumann-Noblesse) integral equation for evaluating potential flow around an offshore structure in waves or a ship that steadily advances through waves or in calm water (presented at the Shanghai 2025 IWWWFB) yields a simplified integral equation that is very well suited for reliable and practical evaluation of potential flows around ships and offshore structures and is incomparably simpler than all known alternative integral equations.

1. Potential flow around a ship that steadily advances through regular waves

Diffraction and radiation of regular waves by a ship that advances at a constant speed V along a straight path through regular waves is considered within the common framework of potential flow theory based on the Green function G that satisfies the classical linear Kelvin-Michell boundary condition at the free-surface plane $\zeta = 0$. The Froude number F and the nondimensional wave frequency f are defined as $F \equiv V/\sqrt{gL}$ and $f \equiv \omega\sqrt{L/g}$ where g is the gravitational acceleration, L denotes the ship length and ω is the encounter frequency of ambient waves. The velocity potential $\widehat{\varphi}(\boldsymbol{\xi}, t)$ of the flow created by the ship is expressed as $\widehat{\varphi}(\boldsymbol{\xi}, t) = \text{Re } \varphi(\boldsymbol{\xi}) e^{-if_\epsilon t}$ where $f_\epsilon \equiv f + i\epsilon$ with $\epsilon = +0$. The ζ axis is vertical and points upward, and the ξ axis lies along the straight path of the ship and points toward the ship bow. The coordinates $\boldsymbol{\xi}$ and \mathbf{x} and the flow potential φ are nondimensional with respect to the ship length L and the acceleration of gravity g . The ship-hull surface and the undisturbed free surface outside the ship are denoted as Σ^H and Σ^F , and Γ and Σ_i^F are the ship waterline and the ship waterplane, i.e. the portion of the free-surface plane $\zeta = 0$ inside Σ^H . The unit vectors $\mathbf{n} \equiv (n^x, n^y, n^z)$ and $\boldsymbol{\nu} \equiv (\nu^x, \nu^y, 0)$ point outside the ship and are normal to the hull surface Σ^H or the waterline Γ of the ship.

2. Basic weakly-singular NN integral equation

The flow potential $\phi \equiv \varphi(\mathbf{x})$ at a point \mathbf{x} of the ship-hull surface Σ^H is determined by the NN integral equation

$$(1 - C^\Gamma) \phi = \phi^H + \phi^{\Sigma^F} \quad \text{where } C^\Gamma \equiv \int_{\Sigma_i^F} d\xi d\eta G_\zeta = - \int_\Gamma dl \boldsymbol{\nu} \cdot \nabla_\xi G^\zeta, \quad (1a)$$

$$\phi^H \equiv \int_{\Sigma^H} da [q^H G + (\phi - \varphi) \mathbf{n} \cdot \nabla_\xi G] \quad \text{and} \quad \phi^{\Sigma^F} \equiv \int_{\Sigma^F} d\xi d\eta [q^F - F p_\xi^F - i f p^F] G \quad (1b)$$

given in [1,2] for offshore structures and in [3-5] for ships. The flux q^H across the body surface Σ^H and the flux q^F and the pressure p^F at the free surface Σ^F are presumed *known* in the NN integral equation (1).

3. Boundary-integral representation of exterior flow

The NN boundary-integral flow representation (1) holds for $\mathbf{x} \in \Sigma^H$ as well as for flow-field points \mathbf{x} located *outside* the body-surface Σ^H where (1) can be verified [4,5] to yield the *exterior* flow representation

$$\varphi(\mathbf{x}) = \int_{\Sigma^H} da(\boldsymbol{\xi}) [q^H(\boldsymbol{\xi}) G(\boldsymbol{\xi}, \mathbf{x}) - \varphi(\boldsymbol{\xi}) \mathbf{n}(\boldsymbol{\xi}) \cdot \nabla_\xi G(\boldsymbol{\xi}, \mathbf{x})] + \int_{\Sigma^F} d\xi d\eta [q^F(\boldsymbol{\xi}) - F p_\xi^F(\boldsymbol{\xi}) - i f p^F(\boldsymbol{\xi})] G(\boldsymbol{\xi}, \mathbf{x}) \quad (2)$$

in which the flow potential $\varphi(\boldsymbol{\xi})$ at $\boldsymbol{\xi} \in \Sigma^H$ is determined by the weakly-singular NN integral equation (1).

4. Rankine-Fourier decomposition of the basic NN integral equation (1)

Application of the fundamental *Rankine-Fourier decomposition* $G = G^R + G^F$, where $G^R \equiv G^R(\boldsymbol{\xi}, \mathbf{x})$ represents the *dominant Rankine singularity* in G and $G^F \equiv G^F(\boldsymbol{\xi}, \mathbf{x})$ is the corresponding Fourier component, in the basic NN integral equation (1) is easily shown to yield [4,5]

$$[1 - C^R(\mathbf{x})] \varphi(\mathbf{x}) = \phi^R(\mathbf{x}) + \phi^F(\mathbf{x}) \quad \text{where } C^R(\mathbf{x}) \equiv \int_{\Sigma_i^F} d\xi d\eta G_\zeta^R = - \int_\Gamma dl(\boldsymbol{\xi}) \boldsymbol{\nu}(\boldsymbol{\xi}) \cdot \nabla_\xi (G^R)^\zeta, \quad (3a)$$

$$\phi^R(\mathbf{x}) \equiv \int_{\Sigma^H} da(\boldsymbol{\xi}) [q^H(\boldsymbol{\xi}) G^R + \{\varphi(\mathbf{x}) - \varphi(\boldsymbol{\xi})\} \mathbf{n}(\boldsymbol{\xi}) \cdot \nabla_\xi G^R] + \int_{\Sigma^F} d\xi d\eta [q^F(\boldsymbol{\xi}) - F p_\xi^F(\boldsymbol{\xi}) - i f p^F(\boldsymbol{\xi})] G^R, \quad (3b)$$

$$\phi^F(\mathbf{x}) \equiv \int_{\Sigma^H} da(\boldsymbol{\xi}) [q^H(\boldsymbol{\xi}) G^F - \varphi(\boldsymbol{\xi}) \mathbf{n}(\boldsymbol{\xi}) \cdot \nabla_\xi G^F] + \int_{\Sigma^F} d\xi d\eta [q^F(\boldsymbol{\xi}) - F p_\xi^F(\boldsymbol{\xi}) - i f p^F(\boldsymbol{\xi})] G^F. \quad (3c)$$

Like the basic NN integral equation (1), the modified NN integral equation (3) holds for an offshore structure in waves as well as a ship that steadily advances through waves or in calm water. G^F in expression (3c) for the Fourier potential ϕ^F is the Fourier component that corresponds to the Rankine component G^R in the Rankine-Fourier decomposition $G = G^R + G^F$.

The function $C^R(\mathbf{x})$ in the modified integral equation (3) only involves Rankine singularities G^R , whereas the function $C^\Gamma(\mathbf{x})$ in the basic NN integral equation (1) involves the *complete* Green function $G \equiv G^R + G^F$. The Rankine potential ϕ^R in the simplified NN integral equation (3) likewise only involves G^R . The function $C^R(\mathbf{x})$ and the Rankine potential $\phi^R(\mathbf{x})$ defined by (3a-b) are now considered.

5. Function $C^R(\mathbf{x})$ and Rankine potential $\phi^R(\mathbf{x})$ in the simplified NN integral equation (3)

The dominant Rankine component G^R in the fundamental Rankine-Fourier decomposition $G = G^R + G^F$ is defined in terms of two elementary Rankine sources $1/r$ and $1/r'$, related to vectors \mathbf{r} and \mathbf{r}' , that are defined as

$$\left\{ \begin{array}{l} \mathbf{r} \equiv (\xi - x, \eta - y, \zeta - z) \\ \mathbf{r}' \equiv (\xi - x, \eta - y, \zeta + z) \end{array} \right\} \text{ and } \left\{ \begin{array}{l} r(\boldsymbol{\xi}, \mathbf{x}) \equiv \sqrt{h^2 + (\zeta - z)^2} \\ r'(\boldsymbol{\xi}, \mathbf{x}) \equiv \sqrt{h^2 + (\zeta + z)^2} \end{array} \right\} \text{ where } h^2 \equiv (\xi - x)^2 + (\eta - y)^2. \quad (4)$$

For an *offshore structure in regular waves*, G^R is given by $4\pi G^R(\boldsymbol{\xi}, \mathbf{x}) = -1/r - 1/r'$. One then has $G_\zeta^R = 0$ at $\zeta = 0$, and expressions (3a-b) become

$$C^R \equiv 0 \text{ and } \phi^R \equiv \int_{\Sigma^H} da \left[\frac{q^H}{4\pi} \left(\frac{-1}{r} - \frac{1}{r'} \right) + \frac{\phi - \varphi}{4\pi} \left(\frac{\mathbf{n} \cdot \mathbf{r}}{r^3} + \frac{\mathbf{n} \cdot \mathbf{r}'}{(r')^3} \right) \right]. \quad (5)$$

For a *ship that steadily advances in calm water or through waves*, one has $4\pi G^R(\boldsymbol{\xi}, \mathbf{x}) = -1/r + 1/r'$ and

$$C^R \equiv \frac{1}{2\pi} \int_{\Gamma} d\ell \frac{(\xi - x)v^x + (\eta - y)v^y}{(\sqrt{h^2 + z^2} - z)\sqrt{h^2 + z^2}} \text{ and } \phi^R \equiv \int_{\Sigma^H} da \left[\frac{q^H}{4\pi} \left(\frac{-1}{r} + \frac{1}{r'} \right) + \frac{\phi - \varphi}{4\pi} \left(\frac{\mathbf{n} \cdot \mathbf{r}}{r^3} - \frac{\mathbf{n} \cdot \mathbf{r}'}{(r')^3} \right) \right]. \quad (6)$$

The Rankine component $4\pi G^R = -1/r + 1/r'$ used in (6) corresponds to an *optimal* Rankine-Fourier decomposition of the Green function relevant to a ship that *steadily advances through waves* [5-7]. The Rankine components in *optimal* Rankine-Fourier decompositions of the Green functions for an offshore structure in waves or a ship that advances in *calm* water, for which one has $F = 0$ or $f = 0$, are defined in [5-7] as

$$4\pi G^R(\boldsymbol{\xi}, \mathbf{x}) = \left\{ \begin{array}{l} -1/r + 1/r' - 2/r^F \\ -1/r - 1/r' + 2/r^f \end{array} \right\} \text{ where } \left\{ \begin{array}{l} r^F \equiv \sqrt{h^2 + (\zeta + z - F^2)^2} \\ r^f \equiv \sqrt{h^2 + (\zeta + z - 1/f^2)^2} \end{array} \right\} \text{ if } \left\{ \begin{array}{l} f = 0 \\ F = 0 \end{array} \right\}.$$

The expressions for the function C^R and the Rankine potential ϕ^R that correspond to these optimal Rankine-Fourier decompositions of the Green functions relevant to a ship that advances in calm water or an offshore structure in waves are trivial minor modifications of expressions (5) and (6) and are given in [4,5].

6. Major features of the weakly-singular NN integral equation (3)

The basic NN integral equation (1) and the related simplified integral equation (3) are based on the *no-flow restriction* $\varphi = 0$ in the infinitesimally thin layer of water above the rigid lid that closes the open body surface Σ^H in the rigid-waterplane flow model [2-5]. This '*dead-water constraint*' is consistent with the fact that the flow around Σ^H does not determine a flow *inside* Σ^H (outside the flow region), which can then be freely specified and in particular may be chosen nil, and arguably precludes flow singularities along the body waterline Γ .

The no-flow restriction $\varphi = 0$ imposed at the body waterplane in the NN flow-model is a '*flow-modeling assumption*' that is not related to the goal of preventing the occurrence of irregular frequencies, and indeed is not sufficient to achieve that goal. Irregular frequencies can be avoided in several ways extensively considered in a broad literature, briefly reviewed in [2,5]. In particular, the condition $\varphi(\mathbf{x}) = 0$ can be imposed at the ship waterplane Σ_i^F in the NN flow-representation (3), which then yields

$$\phi^R(\mathbf{x}) + \phi^F(\mathbf{x}) = [1 - C^R(\mathbf{x})] \varphi(\mathbf{x}) \text{ if } \mathbf{x} \in \Sigma^H \quad \text{and} \quad \phi^R(\mathbf{x}) + \phi^F(\mathbf{x}) = 0 \text{ if } \mathbf{x} \in \Sigma_i^F. \quad (7)$$

The pair of boundary-integral relations (7) applied at $\mathbf{x} \in \Sigma^H$ or at $\mathbf{x} \in \Sigma_i^F$ only involves the potential $\varphi(\boldsymbol{\xi})$ at points $\boldsymbol{\xi} \in \Sigma^H$ and yields an overdetermined system of equations.

7. The NN integral equation (3) is much simpler than all known alternative integral equations

The function $C^R(\mathbf{x})$ in the NN integral equation (3) is defined *explicitly*, i.e. this waterline integral does not involve the flow potential φ . Indeed, *the NN integral equation (3) only involves φ at the ship-hull surface Σ^H* . This integral equation is then incomparably simpler, and much better suited for numerical computations, than the classical Neumann-Kelvin integro-differential equation, which contains a notoriously troublesome waterline integral that involves the flow potential φ and its derivative φ_ξ . For common displacement ships, the NN integral equation (3) only involves distributions of singularities over the ship-hull surface Σ^H . If a *prescribed* distribution of pressure p^F and/or flux q^F is applied at Σ^F , e.g. for surface-effect ships, the NN integral equation (3) involves a distribution of *weak* singularities over a *specified small portion* of the free surface Σ^F around the ship.

The NN integral equation (3) is also incomparably simpler than the alternative integral equations considered in the literature, notably integral equations related to the 'domain-decomposition method' in which the NK

integral equation is applied at a surface that surrounds the body [8], the ‘Rankine-source method’ [9,10] in which the Green function is taken as the basic Rankine source $1/r$, and the ‘double-body-flow linearization method’ in which the nonlinear free-surface boundary condition is linearized about the flow around a ship and its free-surface mirror image [11]. Indeed, these alternative integral equations involve distributions of singularities over the unbounded free surface Σ^F , which must necessarily be truncated. Moreover, the radiation condition must be enforced, in some nontrivial approximate way, in the Rankine-source method. Accurate evaluation of the derivatives of the flow velocity related to the ‘double-body base-flow’, required in the double-body-flow linearization method, likewise is extremely difficult in the vicinity of the ship waterline [12].

8. The NN integral equation (3) is very well suited for reliable and practical numerical solution

The function $C^R(\mathbf{x})$ in the integral equation (3) is expressed in (3a) and in the related expression in (6) as distributions of weakly-singular Rankine singularities along the body waterline Γ . The Rankine potential ϕ^R is similarly defined in (3b) and the related expressions in (5) or (6) by distributions of Rankine sources and *weak* Rankine dipoles over the body surface Σ^H and distributions of Rankine sources over the free surface Σ^F . These distributions of weak Rankine singularities can be accurately evaluated via ordinary integration rules.

The Fourier potential ϕ^F is defined in (3c) by distributions, over the ship-hull surface Σ^H and the free surface Σ^F , of the Fourier component G^F that corresponds to the Rankine component G^R in the decomposition $G = G^R + G^F$. The surface distributions of G^F in (3c) can be evaluated in several ways. In particular, for the simplest case of *wave diffraction-radiation by an offshore structure in deep water*, the Green function and its gradient can be evaluated in a straightforward and efficient manner via global analytical approximations [13-15]. For a *ship that steadily advances in calm deep water*, the non-oscillatory local-flow components contained in G^F and $\nabla_{\xi} G^F$ can be efficiently evaluated via table interpolation [16] or global analytical approximations [17,18] and the wave component in the Fourier potential ϕ^F defined by (3c) can readily be evaluated via the Fourier-Kochin method as in [19] and other studies. Alternative methods for evaluating and integrating G^F and its gradient in the most complicated case of *a ship that steadily advances through regular waves* are given in [20-22]. Thus, the NN integral equation (3) is suited for *reliable* numerical evaluation.

In particular, two important features of the NN integral equation (3) ensure that this integral equation is well suited for numerical solutions via a *low-order* panel method.

(i) A useful feature of the NN integral equation (3) is that it is *weakly* singular. Within the framework of a low-order panel method, in which the flow potential φ is assumed piecewise constant within every panel, this feature implies that the weak dipole $[\varphi(\mathbf{x}) - \varphi(\xi)] \mathbf{n}(\xi) \cdot \nabla_{\xi} G^R$ in (3b), (5) and (6) yields a *nil* contribution of a panel upon itself. Thus, the (relatively complicated) analytical relations for the flow potential due to a uniform dipole distribution over a panel are not required, and indeed are not used in [23]. This numerical study also shows that the errors in computations based on the weakly-singular NN integral equations for $g = \infty$ or $g = 0$ are about half the errors obtained via computations based on the corresponding classical integral equations.

(ii) A crucial feature of the NN integral equation (3) is that it does *not* involve the flow potential φ at the ship waterline Γ . Thus, computations only involve panels (e.g. flat triangles) with *centroids* $\xi_p \equiv (\xi_p, \eta_p, \zeta_p < 0)$ that are submerged below the free-surface plane $\zeta = 0$. As a result, the integrand of the waterline integral that defines the function C^R in (3a) and the related expression (6) is not merely weakly singular but in fact is not singular, and this is also the case for the weak dipoles in the Rankine potential ϕ^R defined by (3b) and the related expressions (5) and (6). These distributions of weakly-singular Rankine singularities can then be evaluated via ordinary integration rules. Another major consequence of the fact that $\zeta_p < 0$ for every panel Σ_p^H that approximates a body surface Σ^H is that one has

$$z + \zeta_p < 0 \text{ and } \exp[k(z + \zeta_p)] \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for } z \leq 0.$$

Thus, one has *exponential decay* of the Fourier integrals related to the Fourier components G^F and $\nabla_{\xi} G^F$ of the Green function G and its gradient in expression (3c) for the Fourier potential ϕ^F in the NN integral equation (3). This feature has a critical influence on the reliability and the robustness of numerical computations for a ship that advances through waves or in calm water, which involve short waves (unlike the trivial case of wave diffraction-radiation by offshore structures).

9. Numerical applications of the NN integral equation (3)

Numerical applications of the NN integral equation (3) are given in [23,4] for the special cases that correspond to the infinite or zero gravity limits $g = \infty$ or $g = 0$. Applications of the integral equation (3) to ships that steadily advance in calm deep water are currently by the author and Prof. Jiayi He. Applications of the integral equation defined by (3a) with (5) and (3c) to diffraction-radiation of regular waves by an offshore structure

in deep water, for which the Green function and its gradient can be easily and efficiently evaluated via the global analytical approximations given in [13-15], are not contemplated by the author but would be valuable. Lastly, applications to a ship that steadily advances through regular waves, for which the Green function is more complicated but can nevertheless be evaluated in several ways [20-22,24], would be especially useful.

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