Combining the nonlinear Froude Krylov + nonlinear hydrostatics loads with second order loads in weakly nonlinear time domain simulations

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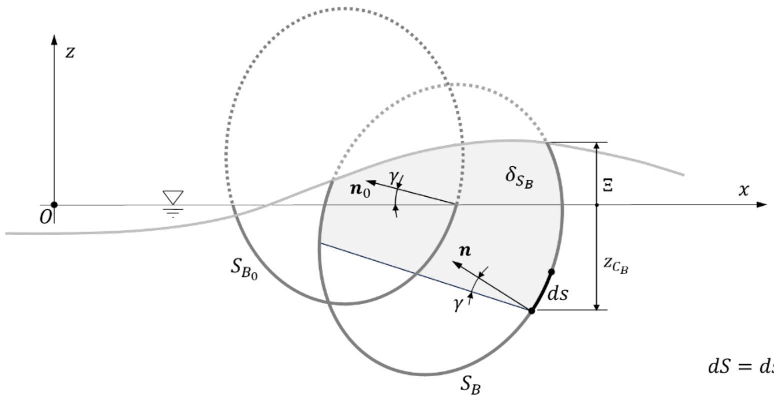
Introduction

The present paper focus on wave induced loading on floating bodies, with the accent on the loads of potential flow origin, within the weakly nonlinear assumptions. The weakly nonlinear wave loading is usually considered using either the second order diffraction-radiation theory or using the so-called nonlinear Froude Krylov + nonlinear hydrostatic model. These two models are usually employed independently but, for some applications (e.g. large body motions in vertical plane) two models need to be combined together. The practical problem which arises when doing that, is that these two types of loading can not be simply added to each other because the second order model implicitly includes the parts of the nonlinear Froude Krylov + nonlinear hydrostatic loading. In other words, there is a risk of double counting of some terms. We refer to Figure 1 where the instantaneous position of the floating body in waves is presented.



$$dS = dsdC = \frac{dz}{\cos\gamma} dC + O(\varepsilon)$$

Figure 1: Instantaneous floating body position in waves.

In order to be fully consistent, the wave loading should be evaluated by integrating the total nonlinear hydrodynamic pressure over the instantaneous wetted body surface $S_B + \delta S_B$. This is very difficult because not only the evaluation of the total hydrodynamic pressure is extremely complex but also the determination of the instantaneous wetted body surface is also complex, and that is why the simplified models are employed in practice. Among those simplified models, the second order model and the nonlinear Froude Krylov + nonlinear hydrostatics model, are the two approaches which are employed most often. The basic features of these models are briefly discussed below.

Fully nonlinear model

The total external loading $\{\boldsymbol{\mathcal{F}}\}$ is composed of the gravity loading $\{\boldsymbol{\mathcal{F}}^g\}$ and the pressure loading $\{\boldsymbol{\mathcal{F}}^h\}$. It is important to note that the gravity force is independent of the body's instantaneous position, and it always acts in the direction of the acceleration of gravity. On the other hand, the direction of the pressure forces is defined by the normal vector on the wetted body surface so that its direction changes when body moves. These two facts have important consequences on the description of the external loading in the respective coordinate systems (earth fixed or body fixed). Here we chose to use the earth fixed coordinate system which is a common practice when rigid body is concerned. The fully nonlinear pressure loads are obtained by integrating the pressure over the instantaneous wetted body surface S_B and we can write:

$$\{\boldsymbol{\mathcal{F}}\} = \{\boldsymbol{\mathcal{F}}^g\} + \{\boldsymbol{\mathcal{F}}^h\} = -mg \begin{Bmatrix} \{\boldsymbol{k}\} \\ \{\boldsymbol{0}\} \end{Bmatrix} + \iint_{S_B} P\{\mathbb{N}\} dS \tag{1}$$

where $\{\mathbb{N}\}$ denotes the generalized normal vector which depends on time, and the body position.
As already indicated, the hydrodynamic part of the problem is solved within the potential flow theory, so that the pressure is given by Bernoulli's equation. For the sake of clarity, the total pressure is decomposed into its dynamic and hydrostatic part:

$$P = -\varrho \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + gz \right] = P^d - \varrho g z \tag{2}$$

and similarly, the corresponding loading is also decomposed in two parts as follows:

$$\{\boldsymbol{\mathcal{F}}^h\} = -\varrho \iint_{S_B} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 \right] \{\mathbb{N}\} dS - \varrho g \iint_{S_B} z \{\mathbb{N}\} dS = \{\boldsymbol{\mathcal{F}}^{hd}\} + \{\boldsymbol{\mathcal{F}}^{hs}\} \tag{3}$$

Second order model

The development of the second order model requires careful linearization procedure where the different quantities are developed into perturbation series and the Taylor series expansion is used in parallel to express the quantities at the instantaneous boundaries as a function of their value at rest. This allows for collecting the quantities at the different orders of magnitude. The procedure is quite heavy and here we summarize the results of interest for the present discussions.
The dynamic pressure component is decomposed as follows:

$$P^{d} = -\varrho\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^{2}\right] = \varepsilon P^{(1)} + \varepsilon^{2}P^{(2)} \tag{4}$$

with:

$$P^{(1)} = -\varrho\frac{\partial\Phi^{(1)}}{\partial t} \quad , \quad P^{(2)} = -\varrho\left[\frac{\partial\Phi^{(2)}}{\partial t} + \frac{1}{2}\left\{\nabla\Phi^{(1)}\right\}^{2} + \left(\left\{\boldsymbol{r}^{(1)}\right\}^{T}\{\nabla\}\right)\frac{\partial\Phi^{(1)}}{\partial t}\right] \tag{5}$$

where $\left\{\boldsymbol{r}^{(1)}\right\}$ is the linear displacement vector of the point attached to the body.
The hydrostatic pressure component is decomposed as follows:

$$-\varrho g z = -\varrho g\{\boldsymbol{r}\}^{\boldsymbol{T}}\{\boldsymbol{k}\} = -\varrho g\left(z^{(0)} + \varepsilon z^{(1)} + \varepsilon^{2}z^{(2)}\right) \tag{6}$$

where:

$$z^{(0)} = z_{G}^{(0)} + z_{0}\ ,\ z^{(1)} = z_{G}^{(1)} + \theta_{x}^{(1)}y_{0} - \theta_{y}^{(1)}x_{0}\ ,\ z^{(2)} = z_{G}^{(2)} + \theta_{x}^{(2)}y_{0} - \theta_{y}^{(2)}x_{0} - \tfrac{1}{2}[\mathcal{H}_{31}x_{0} + \mathcal{H}_{32}y_{0} + \mathcal{H}_{33}z_{0}] \tag{7}$$

with $z^{(0)} = 0$ being the position of the free surface and $\mathcal{H}_{ij}$ denoting the quadratic part of the second order rotations [1].
After integrating the different pressure components over the body, the loading at different orders is obtained as follows:

$$\left\{\boldsymbol{\mathcal{F}}^{hd(1)}\right\} = \iint_{S_{B_0}} P^{(1)}\left\{\mathbb{N}^{(0)}\right\}dS \tag{8}$$

$$\left\{\boldsymbol{\mathcal{F}}^{hd(2)}\right\} = \iint_{S_{B_0}}\left(P^{(2)}\left\{\mathbb{N}^{(0)}\right\} + P^{(1)}\left\{\mathbb{N}^{(1)}\right\}\right)dS + \varrho g\int_{C_{B_0}}\Xi^{(1)}\left(\Xi^{(1)} - z_{C_{B_0}}^{(1)}\right)\frac{\left\{\mathbb{N}^{(0)}\right\}}{\cos\gamma}dC \tag{9}$$

$$\left\{\boldsymbol{\mathcal{F}}^{hs(1)}\right\} = -\varrho g\iint_{S_{B_0}}\left(z^{(1)}\left\{\mathbb{N}^{(0)}\right\} + z^{(0)}\left\{\mathbb{N}^{(1)}\right\}\right)dS \tag{10}$$

$$\left\{\boldsymbol{\mathcal{F}}^{hs(2)}\right\} = -\varrho g\iint_{S_{B_0}}\left(z^{(2)}\left\{\mathbb{N}^{(0)}\right\} + z^{(0)}\left\{\mathbb{N}^{(2)}\right\} + z^{(1)}\left\{\mathbb{N}^{(1)}\right\}\right)dS - \frac{1}{2}\varrho g\int_{C_{B_0}}\left[\left(\Xi^{(1)}\right)^{2} - \left(z_{C_{B_0}}^{(1)}\right)^{2}\right]\frac{\left\{\mathbb{N}^{(0)}\right\}}{\cos\gamma}dC \tag{11}$$

where $\Xi^{(1)}$ is the first order wave elevation (follows from the dynamic free surface condition $P^{(1)}\big|_{z=0} = \varrho g\Xi^{(1)}$) and $\left\{\mathbb{N}^{(i)}\right\}$ is the generalized normal vector of i-th order which follows from the perturbation of its instantaneous value. What is important to note, in the above expressions, is that the surface integration is performed over the mean wetted body surface and an additional line integral over the mean waterline occurs. For the present discussions, it is very important to properly understand the way in which the integral over the perturbed part of the wetted body surface δS_{B} is transformed into the line integral over the waterline C_{B_0}. We refer to Figure 1 and we define the integral of pressure over the instantaneous wetted body surface as:

$$I_{\delta S_B}(p) = \iint_{\delta S_B} p\{\mathbb{N}\}dS = \iint_{\delta S_B} p\{\mathbb{N}\}dsdC = \int_{C_{B_0}}\int_{z_{C_{B_0}}^{(1)}}^{\Xi^{(1)}} p\left\{\mathbb{N}^{(0)}\right\}\frac{dz}{\cos\gamma}dC + O(\varepsilon^{3}) \tag{12}$$

where:

$$z_{C_{B_0}}^{(1)} = z_{G}^{(1)} + \theta_{x}^{(1)}y_{C_{B_0}} - \theta_{y}^{(1)}x_{C_{B_0}} \tag{13}$$

With this in mind, the following hydrodynamic and the hydrostatic parts of the second order loading can be obtained:

$$I_{\delta S_B}\left(P^{(1)}\right) = -\varrho\int_{C_{B_0}}\int_{z_{C_{B_0}}^{(1)}}^{\Xi^{(1)}}\frac{\partial\Phi^{(1)}}{\partial t}\frac{\left\{\mathbb{N}^{(0)}\right\}}{\cos\gamma}dzdC = \varrho g\int_{C_{B_0}}\Xi^{(1)}\left(\Xi^{(1)} - z_{C_{B_0}}^{(1)}\right)\frac{\left\{\mathbb{N}^{(0)}\right\}}{\cos\gamma}dC \tag{14}$$

$$I_{\delta S_B}(-\varrho g z) = -\varrho g\int_{C_{B_0}}\int_{z_{C_{B_0}}^{(1)}}^{\Xi^{(1)}} z\frac{\left\{\mathbb{N}^{(0)}\right\}}{\cos\gamma}dzdC = -\frac{1}{2}\varrho g\int_{C_{B_0}}\left[\left(\Xi^{(1)}\right)^{2} - \left(z_{C_{B_0}}^{(1)}\right)^{2}\right]\frac{\left\{\mathbb{N}^{(0)}\right\}}{\cos\gamma}dC \tag{15}$$

which are exactly the expressions contained in (9) and (11).

At this point, it is also important to note that the velocity potential, which should be used when calculating the pressures $P^{(1,2)}$ and the wave elevation $\Xi^{(1)}$, is the total velocity potential $\Phi^{(1)}$ which includes the incident $\Phi_I^{(1)}$, the diffracted $\Phi_D^{(1)}$ and 6 radiated parts $\Phi_{Rj}^{(1)}$:

$$\Phi^{(1)} = \Phi_I^{(1)} + \Phi_D^{(1)} + \sum_{j=1}^{6} \xi_j \Phi_{Rj}^{(1)} \tag{16}$$

Nonlinear Froude Krylov + nonlinear hydrostatics model

The nonlinear Froude Krylov + nonlinear hydrostatic model is based on rather intuitive correction of the pressure integration close to the free surface. In the first step the intersection of the wave elevation with the body surface is determined where the wave elevation of the incident wave only is considered. This leads to the following pressure distribution in the intermittent region:

$$P^{fk\_hs}(t,z) = P^{fk}(t,z) + P^{hs}(t,z) = \begin{cases} 0 & \text{for} \quad z(t) > \Xi_I(t) \\ P_I(t,0) + \varrho g z(t) & \text{for} \quad 0 < z(t) < \Xi_I(t) \\ P_I(t,z) + \varrho g z(t) & \text{for} \quad z(t) < Min[\Xi_I(t); 0] \end{cases} \tag{17}$$

The pressure distribution is shown on the right part of Figure 2 where it can be observed that the pressure is discontinuous for the negative incident wave elevation. This happens because $P_I(t,z) + \varrho g z(t) \neq 0$ for $z(t) = \Xi_I(t)$. This discontinuity is usually ignored in practice, as well as the fact that only the incident wave is included in the correction, and not the diffracted and radiated pressures which are assumed to be linear. This is of course not fully consistent but …

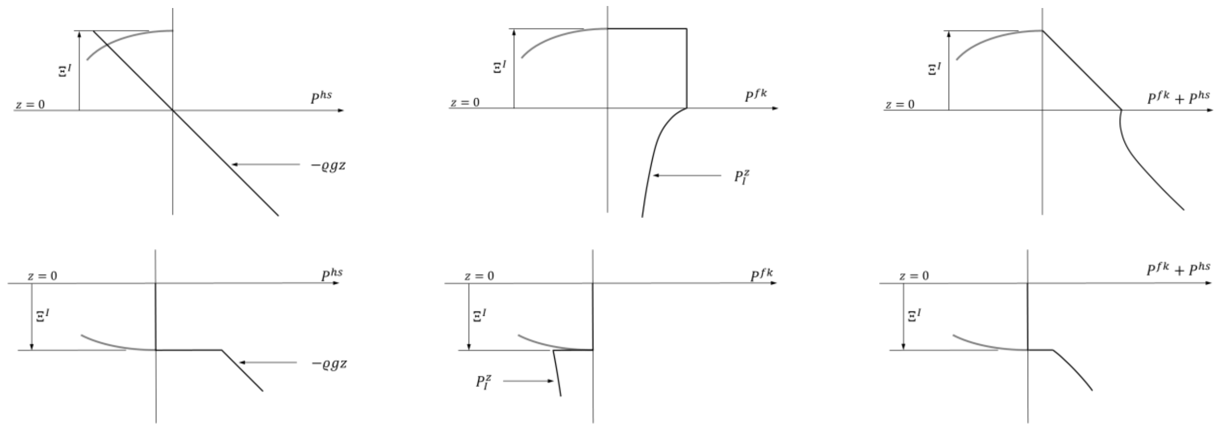


Figure 2: Froude Krylov and nonlinear hydrostatic pressure distribution (Top – positive incident wave elevation, Bottom – negative incident wave elevation)

Once the wetted surface is determined, the Froude Krylov pressure and the hydrostatic pressure P^{fk_hs} are integrated over it, and the resulting loading $\{\boldsymbol{\mathcal{F}}\}^{fk\_hs}$ is obtained:

$$\{\boldsymbol{\mathcal{F}}\}^{fk\_hs} = \iint_{S_B} P^{fk\_hs}\{\mathbb{N}\}dS \tag{18}$$

It is important to note that this loading includes both the zero-th order hydrostatic loading (buoyancy) as well as the first and the second order loadings associated with the linear incident wave and the hydrostatic pressure contributions.

Combined model

From the discussions above, it appears that one of the main differences, between the second order model and the weakly nonlinear Froude Krylov + nonlinear hydrostatic model, are related to the way in which the hydrostatic pressure and the incident wave pressure are applied on the body. In practice this means that, when combining (summing up) the two models, we should remove the hydrostatic and the incident wave pressures from the second order loading. We formally write:

$$\begin{aligned} \{\boldsymbol{\mathcal{F}}\} = \; & \{\boldsymbol{\mathcal{F}}^g\} + \{\boldsymbol{\mathcal{F}}\}^{fk\_hs} + \varepsilon\left(\left\{\boldsymbol{\mathcal{F}}^{hs(1)}\right\} - \left\{\boldsymbol{\mathcal{F}}^{hs(1)}\right\}^{fk\_hs} + \left\{\boldsymbol{\mathcal{F}}^{hd(1)}\right\} - \left\{\boldsymbol{\mathcal{F}}^{hd(1)}\right\}^{fk\_hs}\right) \\ & + \varepsilon^2\left(\left\{\boldsymbol{\mathcal{F}}^{hs(2)}\right\} - \left\{\boldsymbol{\mathcal{F}}^{hs(2)}\right\}^{fk\_hs} + \left\{\boldsymbol{\mathcal{F}}^{hd(2)}\right\} - \left\{\boldsymbol{\mathcal{F}}^{hd(2)}\right\}^{fk\_hs}\right) \end{aligned} \tag{19}$$

where the notation $\left\{\boldsymbol{\mathcal{F}}^{hi(j)}\right\}^{fk\_hs}$ is used to denote the quantities in (8),(9),(11) and (11) which are already included in the nonlinear Froude Krylov + nonlinear hydrostatic loading $\{\boldsymbol{\mathcal{F}}\}^{fk\_hs}$:

$$\{\boldsymbol{\mathcal{F}}^{hd(1)}\}^{fk\_hs} = \iint_{S_{B_0}} P_I^{(1)}\{\mathbb{N}^{(0)}\}dS \tag{20}$$

$$\{\boldsymbol{\mathcal{F}}^{hd(2)}\}^{fk\_hs} = \iint_{S_{B_0}} \left(P_{I\_T}^{(2)}\{\mathbb{N}^{(0)}\} + P_I^{(1)}\{\mathbb{N}^{(1)}\}\right)dS + \varrho g \int_{C_{B_0}} \Xi_I^{(1)}\left(\Xi_I^{(1)} - z_{C_{B_0}}^{(1)}\right)\frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma}dC \tag{21}$$

$$\{\boldsymbol{\mathcal{F}}^{hs(1)}\}^{fk\_hs} = -\varrho g \iint_{S_{B_0}} \left(z^{(1)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(1)}\}\right)dS \tag{22}$$

$$\{\boldsymbol{\mathcal{F}}^{hs(2)}\}^{fk\_hs} = -\varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(2)}\} + z^{(1)}\{\mathbb{N}^{(1)}\}\right)dS - \frac{1}{2}\varrho g \int_{C_{B_0}} \left[\left(\Xi_I^{(1)}\right)^2 - \left(z_{C_{B_0}}^{(1)}\right)^2\right]\frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma}dC \tag{23}$$

The linear incident wave pressure $P_I^{(1)}$ and the linear incident wave elevation $\Xi_I^{(1)}$ are obtained by simply replacing the total velocity potential $\Phi^{(1)}$ with its incident part $\Phi_I^{(1)}$ (see eqn. (16)) in the corresponding expressions, but the term $P_{I\_T}^{(2)}$, which occurs in (21), requires particular explanation. This term arises from the Taylor series expansion of the linear incident wave potential $\Phi_I^{(1)}$ and is given by:

$$P_{I\_T}^{(2)} = -\varrho\left(\{\boldsymbol{r}^{(1)}\}^T\{\nabla\}\right)\frac{\partial\Phi_I^{(1)}}{\partial t} \tag{24}$$

Since, by definition, the nonlinear Froude Krylov + nonlinear hydrostatics model includes the integration of the linear incident wave pressure at the instantaneous position of the body, it follows that this term needs to be removed from the second order loading. In summary the modified second order loading, which we denote by $\{\boldsymbol{\mathcal{F}}^{hi(j)}\}^*$ and which needs to be combined (summed up) with the nonlinear Froude Krylov + nonlinear hydrostatics loading is given by:

$$\{\boldsymbol{\mathcal{F}}^{hi(j)}\}^* = \{\boldsymbol{\mathcal{F}}^{hi(j)}\} - \{\boldsymbol{\mathcal{F}}^{hi(j)}\}^{fk\_hs} \quad \text{, for } i = s,d \text{ and } j = 1,2 \tag{25}$$

In practice, this means that the classical diffraction-radiation codes will have to be slightly modified by excluding the terms (20), (22), (21) and (23), which is relatively straightforward.

Discussions

The need for weakly nonlinear load models (second order or Froude Krylov + nonlinear hydrostatics) occurs in the context of the global performance analysis of the floating systems whose natural frequencies are lower than the wave frequencies which are contained in the wave spectrum. Typical examples are the moored floating systems of any type (FPSO, semi submersibles, TLP's …) for which the lack of the hydrostatic stiffness leads to the large natural periods (low frequencies) in the horizontal modes of motion (surge, sway & yaw). The linear theory not being able to provide any excitation at those frequencies, the need for weakly nonlinear loading models occurred, and the second order models, which are able to produce the loading at lower frequencies $(\omega_i - \omega_j)$, emerged as the reliable practical tool. These models are used with success in practical engineering software for global performance analysis. Within these models, the usual practice is to evaluate the second order loads in frequency domain and to use them in the time domain by simple reconstruction of the frequency domain data. This approach works well in many cases as far as the vertical motions remain limited i.e. close to linear. The problem arises when the vertical motions become large so that the changes of the wetted surface close to waterline becomes significant. In those cases, the second order loads (QTF's), which are pre-calculated using the linear motions (RAO's), might become inaccurate and there is a need for more appropriate time domain models. One of those time domain models, still based on using the precalculated linear frequency domain database (excluding the motions), was proposed in [3] where the Cummins type approach [1], both at first and second order, was introduced. We believe that this approach has a good potential in practice but this needs to be confirmed by the comparisons with the experience (model tests or full-scale measurements). This is left for further work and only limited validation results will be presented at the Workshop.

References

[1] Cummins W.E., 1962.: "The impulse response function and the ship motions.", Schiffsctechnik, Vol. 47

[2] Malenica S. 2023.: "Second-order theory of wave body interactions. Comparisons of the different formulations for rigid body." BV Internal document

[3] Malenica S., Blanc A., Jagite G. & De Hauteclocque G., 2025 : " Practical evaluation of the second order loads in time domain ", IWWWFB 2025, Shanghai, China.

[4] Molin B., 2023.: "Offshore Structure Hydrodynamics", Cambridge University Press.