A Newly Proposed Method of Directly Calculating the 3-D Time-Domain Green Function of Infinite Depth with High Accuracy and Efficiency

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HIGHLIGHTS

An accurate and fast method for calculating the 3-D Green function and its derivatives directly in the time domain is given newly.

1 INTRODUCTION

The time-domain simulation with the potential theory is needed for the direct stability assessment in the second-generation intact stability criteria. 3-D time-domain Green function is a crucial factor. Wehausen and Laitone(1960) gave an integration of the 3-D time-domain free-surface Green function. Liapis (1986) and King (1987) proposed a solution in five areas, while the accuracy of each area's boundary was inconsistent. Newman (1985, 1990) also proposed a solution for the 3-D time-domain free-surface Green function, but its derivatives have not published. Liapis (1986) proposed the Dawson method for the Green function without further integration formulas. Dai and Pan (2019) further gave the formulas of Dawson Integration for the Green function. Clement (1998) proposed the fourth-order ordinary differential equation method (ODE) for the Green function, while the method couldn't be used to calculate the Green function directly in the time domain. There is no published method for numerically calculating the 3-D free surface Green function directly in the time domain, except for some interpolation methods and fitting coefficient methods. We newly give a more accurate and faster method for calculating the 3-D Green function directly in the time domain with a new formula when the spatial parameter μ is approaching 1.

2. 3-D TIME-DOMAIN FREE-SURFACE GREEN FUNCTION AND ITS DERIVATIVES

The integration of the 3-D time-domain Green function, as shown in Eq. (1), is used by Liapis (1986), King (1987), Newman(1990), and Huang(1992). The Green function represents the potential at the field point $P(x,y,z)$ and at time t due to an impulsive source at the point $Q(\xi,\eta,\zeta)$ suddenly created and annihilated at time τ. U_0 is constant forward speed. This $G_0(P,Q)$ can be solved by the method proposed by Hess and Smith (1964). Here, we discuss the calculating method on $\tilde{G}(P,Q;t-\tau)$ and its derivatives.

This $\tilde{G}(P,Q;t-\tau)$ can be expressed by the following formula.

$$\tilde{G}(P,Q,t-\tau)=2\sqrt{g}r'^{-3/2}\cdot G_1;\quad G_1=\int_0^\infty \sqrt{\lambda}\sin\left(\sqrt{\lambda}\cdot\beta\right)e^{-\lambda u}J_0(\lambda\sqrt{1-\mu^2})\,\mathrm{d}\lambda \tag{1}$$

$$\begin{aligned}&\lambda=kr';\mu=-(z+\zeta)/r'=1/\sqrt{1+R^2/(z+\zeta)^2}=\cos\theta\in[0,1]\\ &r'=\sqrt{\left((x-\xi+U_0(t-\tau)\right)^2+\left(y-\eta\right)^2+\left(z+\zeta\right)^2},\sqrt{1-\mu^2}=R/r'=\sin\theta\end{aligned} \tag{2}$$

The derivatives can be obtained as the same as those in the reference (King, 1987). $\hat{G}(\mu,\beta)$ is used in reference (King, 1987), and here $\hat{G}(\mu,\beta)=2G_1$.

The authors derived the 3-D time-domain Green function's derivatives as follows.

$$\frac{\partial G_1}{\partial \beta} = G_2 = \int_0^{\infty} \lambda \cos\left(\sqrt{\lambda}\cdot\beta\right) e^{-\lambda\mu} J_0(\lambda\sqrt{1-\mu^2})\,\mathrm{d}\lambda \tag{3}$$

$$\frac{\partial G_1}{\partial \mu} = \frac{1.5\mu\cdot G_1 + 0.5\beta\mu\cdot G_2 - G_3}{1-\mu^2} \quad (0 \le u < 1) \tag{4}$$

We define the function $F(p)$ as follows,

$$F(p) = \int_0^{\infty} \lambda^p e^{\mathrm{i}\sqrt{\lambda}\cdot\beta} e^{-\lambda\mu} J_0(\lambda\sin\theta)\,\mathrm{d}\lambda \tag{5}$$

The G_1, G_2, G_3 can be expressed as follows.

$$G_1 = \mathrm{Im}\left[F(\frac{1}{2})\right];\quad G_2 = \mathrm{Re}\left[F(1)\right];\quad G_3 = \mathrm{Im}\left[F(\frac{3}{2})\right] \tag{6}$$

Therefore, the Green function $\tilde{G}(P,Q;t-\tau)$ and its derivatives can be obtained by solving the G_1, G_2, G_3.

2.1 Series Expansion Method

When the time parameter β is small, we derived the following formulas to solve the G_1, G_2, G_3.

$$G_1 = \mathrm{Im}\left[F(\frac{1}{2})\right] = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{2m+1}}{(2m+1)!}(m+1)!P_{m+1}(\mu) \tag{7}$$

$$G_2 = \mathrm{Re}\left[F(1)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{2m}}{(2m)!}(m+1)!P_{m+1}(\mu) \tag{8}$$

$$G_3 = \mathrm{Im}\left[F(\frac{3}{2})\right] = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{2m+1}}{(2m+1)!}(m+2)!P_{m+2}(\mu) \tag{9}$$

$$P_0(\mu) = 1,\quad P_1(\mu) = \mu,\quad P_m(\mu) = \frac{(2m-1)\mu P_{m-1}(\mu) - (m-1)P_{m-2}(\mu)}{m} \quad (m \ge 2) \tag{10}$$

2.2 Asymptotic Expansion Method

When the time parameter β is large, we derived the following formulas to solve the G_1, G_2, G_3.

$$F(p) = 2f_0(p) + f_2(p) \tag{11}$$

$$f_0(p) \approx \left(\frac{\mathrm{i}}{\beta}\right)^{2p+2} \sum_{n=0}^{\infty} \frac{P_n(u)\Gamma(2p+2+2n)}{n!\beta^{2n}} \tag{12}$$

$$f_2\left(\frac{1}{2}\right) = \omega_2 \sqrt{\frac{2}{\sin\theta}}\, \mathrm{e}^{\frac{-1}{4}\beta^2 e^{-\mathrm{i}\theta} - \frac{\mathrm{i}}{2}\theta + \frac{\mathrm{i}\pi}{4}} \left\{ \left[\sum_{n=0}^{5} C_n \left(\frac{\mathrm{i}}{\sin\theta}\right)^n \omega_2^{-2n} \cdot \left(1 + \sum_{m=1}^{5-n} d_{nm}\left(\frac{1}{2}\right) \omega_2^{-2m} e^{-\mathrm{i}\theta m}\right)\right] + C_0 \right\} \tag{13}$$

$$f_2(1) = \omega_2^2 \sqrt{\frac{2}{\sin\theta}}\, \mathrm{e}^{\frac{-1}{4}\beta^2 e^{-\mathrm{i}\theta} - \frac{\mathrm{i}}{2}\theta + \frac{\mathrm{i}\pi}{4}} \left\{ \left[\sum_{n=2}^{5} C_n \left(\frac{\mathrm{i}}{\sin\theta}\right)^n \omega_2^{-2n} \cdot \left(1 + \sum_{m=1}^{5-n} d_{nm}(1) \omega_2^{-2m} e^{-\mathrm{i}\theta m}\right)\right] + C_1 \frac{\mathrm{i}\omega_2^{-2}}{\sin\theta} + C_0(1 - \frac{2}{\beta^2} e^{\mathrm{i}\theta}) \right\} \tag{14}$$

$$f_2\left(\frac{3}{2}\right) = \omega_2^3 \sqrt{\frac{2}{\sin\theta}}\, \mathrm{e}^{\frac{-1}{4}\beta^2 e^{-\mathrm{i}\theta} - \frac{\mathrm{i}}{2}\theta + \frac{\mathrm{i}\pi}{4}} \left\{ \left[\sum_{n=2}^{5} C_n \left(\frac{\mathrm{i}}{\sin\theta}\right)^n \omega_2^{-2n} \cdot \left(1 + \sum_{m=1}^{5-n} d_{nm}\left(\frac{3}{2}\right) \omega_2^{-2m} e^{-\mathrm{i}\theta m}\right)\right] + C_1 \frac{\mathrm{i}\omega_2^{-2}}{\sin\theta} + C_0(1 - \frac{6}{\beta^2} e^{\mathrm{i}\theta}) \right\} \tag{15}$$

$$C_n = \frac{(2n-1)!!(2n-1)!!}{8^n n!}; \quad d_{nm}(p) = \frac{(2m+2n-2p-1)!}{(2n-2p-1)!m!2^{2m}} \tag{16}$$

2.3 Dawson Integration Method

We derived the following formulas of Dawson Integration to solve G_1, G_2, G_3.

$$G_1 = \mathrm{Re}\left\{ \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{1}{a^3} \mathrm{Daw}\left(\frac{\beta}{2a}\right) + \frac{\beta}{2a^4} - \frac{\beta^2}{2a^5} \mathrm{Daw}\left(\frac{\beta}{2a}\right) \right] \mathrm{d}\phi \right\} \tag{17}$$

$$G_2 = \mathrm{Re}\left\{ \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{1}{a^4} - \frac{3\beta}{2a^5} \mathrm{Daw}\left(\frac{\beta}{2a}\right) - \frac{\beta}{4a^6} + \frac{\beta^3}{4a^7} \mathrm{Daw}\left(\frac{\beta}{2a}\right) \right] \mathrm{d}\phi \right\} \tag{18}$$

$$G_3 = \mathrm{Re}\left\{ \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{3}{2a^5} \mathrm{Daw}\left(\frac{\beta}{2a}\right) + \frac{5\beta}{4a^6} - \frac{3\beta^2}{2a^7} \mathrm{Daw}\left(\frac{\beta}{2a}\right) - \frac{\beta^3}{8a^8} + \frac{\beta^4}{8a^9} \mathrm{Daw}\left(\frac{\beta}{2a}\right) \right] \mathrm{d}\phi \right\} \tag{19}$$

$$a^2 = \mu - \mathrm{i}\sqrt{1-\mu^2}\cos\phi; \quad \mathrm{Daw}(z) = e^{-z^2} \int_0^z e^{t^2} dt \tag{20}$$

2.4 A New Formula When μ Is Approaching 1

The above formula does not consider the situation when the spatial parameter μ is approaching 1. We derived a new formula when the spatial parameter μ is approaching 1.

$$\frac{\partial G_1}{\partial \mu} = -G_3 + \frac{\mu}{2} \mathrm{Re}(G_{40} + G_{42}) \quad (\mu \approx 1) \tag{21}$$

$$G_{40} = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{15}{4a^4} - \frac{3\beta^2}{8a^6} \right) \left(\frac{1}{a^3} \mathrm{Daw}(\frac{\beta}{2a}) + \frac{\beta}{2a^4} - \frac{\beta^2}{2a^5} \mathrm{Daw}(\frac{\beta}{2a}) \right) + \left(\frac{9\beta}{4a^4} - \frac{\beta^3}{8a^6} \right) \left(\frac{1}{a^5} - \frac{5\beta}{2a^5} \mathrm{Daw}(\frac{\beta}{2a}) - \frac{\beta^2}{4a^6} + \frac{\beta^3}{4a^7} \mathrm{Daw}(\frac{\beta}{2a}) \right) \mathrm{d}\phi$$

$$G_{42} = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left\{ \left(\frac{15}{4a^4} - \frac{3\beta^2}{8a^6} \right) \left(\frac{1}{a^3} \mathrm{Daw}(\frac{\beta}{2a}) + \frac{\beta}{2a^4} - \frac{\beta^2}{2a^5} \mathrm{Daw}(\frac{\beta}{2a}) \right) + \left(\frac{9\beta}{4a^4} - \frac{\beta^3}{8a^6} \right) \left(\frac{1}{a^5} - \frac{3\beta}{2a^5} \mathrm{Daw}(\frac{\beta}{2a}) - \frac{\beta^2}{4a^6} + \frac{\beta^3}{4a^7} \mathrm{Daw}(\frac{\beta}{2a}) \right) \right\} \cos 2\phi \, \mathrm{d}\phi \tag{22}$$

$$\frac{\partial G_1}{\partial \mu} = \mathrm{Re}\left\{ -\left[\frac{3}{2} \mathrm{Daw}\left(\frac{\beta}{2}\right) + \frac{5\beta}{4} - \frac{3\beta^2}{2} \mathrm{Daw}\left(\frac{\beta}{2}\right) - \frac{\beta^3}{8} + \frac{\beta^4}{8} \mathrm{Daw}\left(\frac{\beta}{2}\right) \right] \right\}$$
$$+ \mathrm{Re}\left\{ \frac{1}{2} \left[\left(\frac{15}{4} - \frac{3\beta^2}{8} \right) \left(\mathrm{Daw}(\frac{\beta}{2}) + \frac{\beta}{2} - \frac{\beta^2}{2} \mathrm{Daw}(\frac{\beta}{2}) \right) + \left(\frac{9\beta}{4} - \frac{\beta^3}{8} \right) \left(1 - \frac{3\beta}{2} \mathrm{Daw}(\frac{\beta}{2}) - \frac{\beta^2}{4} + \frac{\beta^3}{4} \mathrm{Daw}(\frac{\beta}{2}) \right) \right] \right\} (\mu = 1) \tag{23}$$

3 RESULTS

The newly proposed method of directly calculating the 3-D time-domain Green function and its derivatives with high accuracy and efficiency, as shown in Fig.1.Series Expansion is used when $0 \le \beta \le 8.5, 0 \le \mu \le 0.40$, $0 \le \beta \le 7.2, 0.40 < \mu \le 0.99$; Asymptotic Expansion method is used when $8.5 < \beta, 0 \le \mu \le 0.40$, $10.1 < \beta, 0.40 < \mu \le 0.99$;Dawson Integration method is used in the rest regions. Nondimensional Green Function and its derivatives are shown in Fig.2. A 10-digit accuracy in general and a 6-digit accuracy at the boundary of each method can be obtained.

We check the time cost of our proposal. When $\mu = 0.5$, and the time parameter β is 0 to 40 with a step 0.1, 7 ms is cost to calculate the Green function and its derivatives, that is too say, 17.5 µs is cost to calculate one Green function and its corresponding derivatives. The general computer is used with the CPU Intel Core i7-9700, 3.00GHZ. Both the interpolation methods with a pre-calculated database and the fitting coefficient methods with a pre-calculated database will reduce their accuracy, and the interpolation methods for the 3-D time-domain Green function are also very complicated. According to the published paper on the 3-D time-domain Green function, we believe that our method mentioned in this paper is an accurate and fast method for calculating the 3-D Green function and derivatives directly in the time domain at this stage.

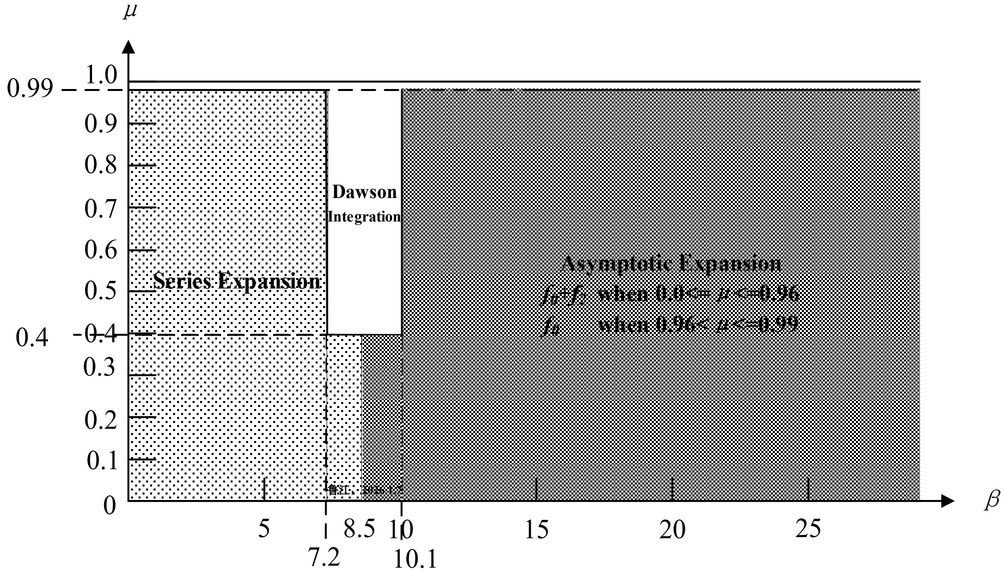

Figure 1: Regions for Green function and its derivatives' evaluations

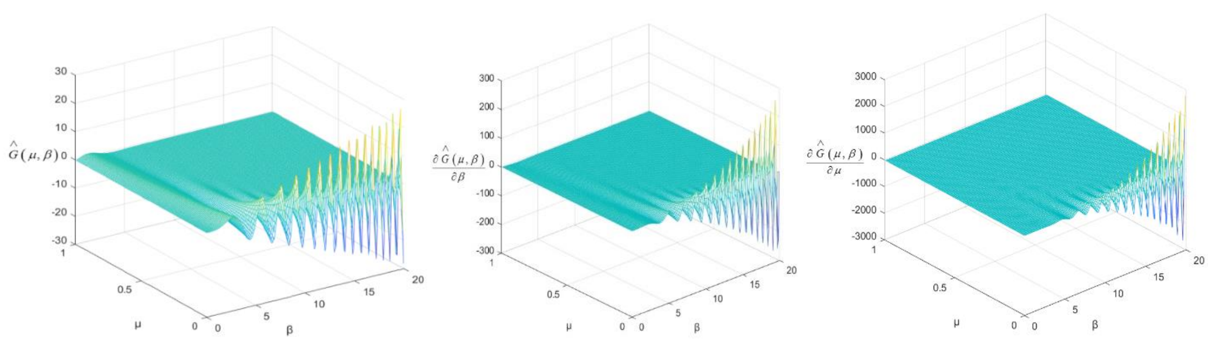

Figure 2: Nondimensional Green Function and its derivatives

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