Relationships of Kochin Function with Eigenfunction-Expansion Coefficients in Multiple-Body Wave Interactions

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1 Introduction

This paper is concerned with hydrodynamic relationships in the multiple floating-body interaction problem, typically the energy-conservation law between the wave-damping force and the radiation wave due to forced body motions, and the Haskind-Newman relation between the wave-exciting force and the radiation wave.

For multiple floating bodies, computations can be made with both boundary element method (BEM) and eigenfunction-expansion method (EFEM). The BEM treats all the bodies as a single large floating body, and hence hydrodynamic relations in terms of the Kochin function, which is the complex amplitude of far-field outgoing waves generated by the body disturbance, can be applied in the same way as for a single floating body. Studies using EFEM on the Haskind relation and the energy-conservation law have been made by Zhong & Yeung [1] and Fabregas Flavià & Clément [2], but the relationship of the coefficients of eigenfunction expansions with the Kochin function is not fully explained. Furthermore, the control surface used in Green's formula to derive the Haskind relation was taken only at a sufficiently large distance surrounding all floating bodies. However, in multiple floating bodies, Green's formula is also applicable to a cylindrical surface that encompasses only the floating body under consideration for computing hydrodynamic forces.

2 Formulation and Computation Methods

2.1 Velocity potentials and boundary conditions

With the assumption of incompressible and inviscid fluid with irrotational motion, the velocity potential is introduced, and a linearized boundary-value problem in the frequency domain is considered, with time-dependent term $e^{i\omega t}$ excluded. Then, the spatial part of the velocity potential is expressed in the form

$$\phi(\boldsymbol{x}) = \frac{g\zeta_a}{i\omega}\Big\{\,\phi_I(\boldsymbol{x}) + \phi_B(\boldsymbol{x})\,\Big\} \tag{1}$$

$$\phi_B(\boldsymbol{x}) = \phi_S(\boldsymbol{x}) - K\sum_{\ell=1}^{N_B}\sum_{j=1}^{6} X_j^\ell\,\phi_j^\ell(\boldsymbol{x}), \quad \phi_I(\boldsymbol{x}) + \phi_S(\boldsymbol{x}) \equiv \phi_D(\boldsymbol{x}) \tag{2}$$

$$\phi_I(\boldsymbol{x}) = Z_0(z)\,e^{-ik_0(x\cos\beta + y\sin\beta)} = \sum_{m=-\infty}^{\infty}(-i)^m e^{im\beta}\Big\{Z_0(z)\,J_m(k_0 r)\,e^{-im\theta}\Big\} \tag{3}$$

where

$$Z_0(z) = \frac{\cosh k_0(z-h)}{\cosh k_0 h}, \quad k_0\tanh k_0 h = \frac{\omega^2}{g} = K \tag{4}$$

Here, ζ_a, ω, k_0, and β are the amplitude, circular frequency, wavenumber, and propagation angle of the incident wave, respectively, g is the gravitational acceleration; these are correlated with the dispersion relation shown in (4). ϕ_I and ϕ_S denote the incident-wave and scattering potentials and their sum ϕ_D is referred to as the diffraction potential. ϕ_B denotes the body-disturbance potential, in which ϕ_j^ℓ is the radiation potential due to the j-th ($= 1 \sim 6$) mode of motion of floating body ℓ ($= 1 \sim N_B$), with complex amplitude X_j^ℓ (which is normalized with ζ_a) and each of N_B multiple floating bodies supposed to oscillate independently.

As shown in Fig. 1, the positive z-axis is taken vertically downward, with $z = 0$ and h being still water surface and constant water bottom, respectively. In addition to the space-fixed global Cartesian coordinates (x, y, z), the cylindrical coordinates (r, θ, z) will be used, with $x + iy = r\,e^{i\theta}$, and also the local coordinates $x_k + iy_k = r_k\,e^{i\theta_k}$ ($k = 1 \sim N_B$) with the origin at the center of each floating body will be used.

Explicit forms of the linearized boundary conditions are well-known and hence omitted here, except for the body boundary condition which is written as

$$[\,H\,] \quad \frac{\partial\phi_D}{\partial n} = 0, \qquad \frac{\partial\phi_j^\ell}{\partial n} = n_j^\ell\,\delta_{k\ell} \quad \text{on } S_k \text{ and } j = 1 \sim 6 \tag{5}$$

where S_k denotes the submerged surface of floating body k, and n_j^ℓ the j-th component of the outward normal vector $\boldsymbol{n}^\ell$ of floating body ℓ (defined as $\boldsymbol{x}^\ell \times \boldsymbol{n}^\ell$ for $j = 4 \sim 6$). Since only floating body ℓ is supposed to oscillate, the right-hand side of (5) is zero for $k \neq \ell$; which is indicated with Kronecker's delta $\delta_{k\ell}$.

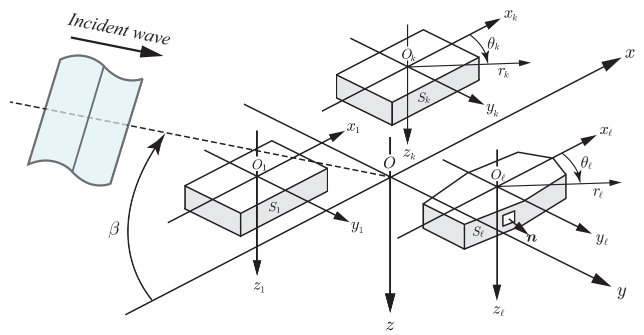

Fig. 1 Coordinate systems and notations

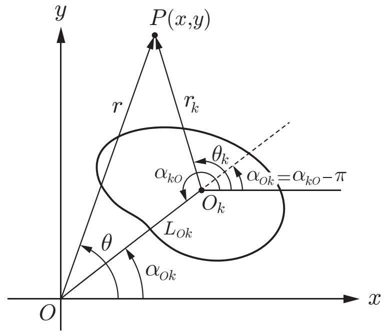

Fig. 2 Relation of global and local coordinates

2.2 Boundary element method

Regarding all floating bodies as a single large body, the boundary element method can be applied. By solving the integral equation for the velocity potential on the submerged surface of each floating body, we can obtain the velocity potential at any field point, $P = (x, y, z)$, in the fluid domain. Writing only the radiation potential, it is given as

$$\phi_j^\ell(P) = \sum_{k=1}^{N_B} \iint_{S_k} \left\{ \frac{\partial \phi_j^\ell(Q)}{\partial n_Q} - \phi_j^\ell(Q) \frac{\partial}{\partial n_Q} \right\} G(P;Q)\, dS_Q \tag{6}$$

where $Q = (x', y', z')$ denotes the integration point on each floating-body surface in the global coordinate system. $G(P;Q)$ is the free-surface Green function for constant finite water depth, which consists of two distinctly different components; one is the outward propagating wave, typically represented by the second kind Hankel function $H_0^{(2)}(k_0R)$, and the other is the evanescent wave.

In a far field ($r \gg 1$), evanescent waves decay and the asymptotic expression for $H_0^{(2)}(k_0R)$ can be used, with R the distance between P and Q in the horizontal plane approximated as $R \simeq r - (x' \cos\theta + y' \sin\theta)$. With the same transformation as for a single floating body, we can obtain the far-field asymptotic expression of $\phi_B(\boldsymbol{x})$ in the form

$$\phi_B(\boldsymbol{x}) \simeq \frac{i}{2} C_0 \sqrt{\frac{2}{\pi k_0 r}} \Big\{ H(k_0, \theta) \Big\} Z_0(z)\, e^{-ik_0 r + i\pi/4}, \quad C_0 = \frac{k_0^2}{K + h(k_0^2 - K^2)} \tag{7}$$

where

$$H(k_0, \theta) = H_S(k_0, \theta) - K \sum_{\ell=1}^{N_B} \sum_{j=1}^{6} X_j^\ell\, H_j^\ell(k_0, \theta) \tag{8}$$

$$H_j^\ell(k_0, \theta) = \sum_{k=1}^{N_B} \iint_{S_k} \left\{ \frac{\partial \phi_j^\ell}{\partial n} - \phi_j^\ell \frac{\partial}{\partial n} \right\} Z_0(z')\, e^{ik_0(x'\cos\theta + y'\sin\theta)}\, dS \tag{9}$$

is defined as the Kochin function in the multiple floating-body problem, representing the θ-dependency of the disturbance wave. (Here, the equation for $H_S(k_0, \theta)$ is omitted.) This θ-dependency can be expressed normally as the Fourier series, which can be readily obtained by using the relation shown in (3). That is,

$$e^{ik_0(x'\cos\theta + y'\sin\theta)} = \sum_{n=-\infty}^{\infty} (i)^n \Big\{ J_n(k_0 r')\, e^{in\theta'} \Big\} e^{-in\theta} \tag{10}$$

Noting that $x' + iy' = r' e^{i\theta'}$ is the global coordinates, we transform further (10) into the local coordinates (r_k', θ_k') of floating body k; which can be realized using the following addition theorem

$$J_n(k_0 r')\, e^{in\theta'} = \sum_{p=-\infty}^{\infty} J_{n-p}(k_0 L_{ok})\, e^{i(n-p)\alpha_{ok}} \Big\{ J_p(k_0 r_k')\, e^{ip\theta_k'} \Big\} \tag{11}$$

where $L_{ok}\, e^{i\alpha_{ok}}$ denotes the position vector of the origin of local coordinates of floating body k (see Fig. 2).

Summing up these, we can obtain the Fourier-series expansion of the Kochin function in the form

$$H(k_0, \theta) = \sum_{n=-\infty}^{\infty} c_n\, e^{-in\theta}, \quad c_n = (i)^n \left\{ c_{Sn} - K \sum_{\ell=1}^{N_B} \sum_{j=1}^{6} X_j^\ell c_{jn}^\ell \right\} \tag{12}$$

where

$$\left\{ \begin{array}{c} c_{Sn} \\ c_{jn}^\ell \end{array} \right\} = \sum_{k=1}^{N_B} \sum_{p=-\infty}^{\infty} J_{n-p}(k_0 L_{ok})\, e^{i(n-p)\alpha_{ok}} \left\{ \begin{array}{c} c_{Sp}^k \\ c_{jp}^{\ell k} \end{array} \right\} \tag{13}$$

$$c_{jp}^{\ell k} = \iint_{S_k} \left\{ \frac{\partial \phi_j^\ell}{\partial n} - \phi_j^\ell \frac{\partial}{\partial n} \right\} Z_0(z')\, J_p(k_0 r_k')\, e^{ip\theta_k'}\, dS \tag{14}$$

Equation (14) is written using the local coordinates of each floating body ($k = 1 \sim N_B$) and (12)–(13) provide a formula for the complex Fourier coefficients in the global coordinate system for multiple floating bodies.

2.3 Eigenfunction expansion method

Using the eigenfunction expansions in the cylindrical coordinate system, the disturbance potential $\phi_B(\boldsymbol{x})$ due to multiple floating bodies can be expressed in a vector form as the summation of each body disturbance as follows:

$$\phi_B(\boldsymbol{x}) = \sum_{k=1}^{N_B} \left(\left\{ A_S^k \right\}^T - K \sum_{\ell=1}^{N_B} \sum_{j=1}^{6} X_j^\ell \left\{ \mathcal{R}_j^{\ell k} \right\}^T \right) \left\{ \psi_S^k \right\} \equiv \sum_{k=1}^{N_B} \left\{ \mathcal{A}_B^k \right\}^T \left\{ \psi_S^k \right\} \tag{15}$$

where

$$\left\{ \psi_S^k \right\} = \left\{ \begin{array}{c} Z_0(z) H_p^{(2)}(k_0 r_k)\, e^{-ip\theta_k} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Z_q(z) K_p(k_q r_k)\, e^{-ip\theta_k} \end{array} \right\} \quad \begin{array}{l} p = 0, \pm 1, \pm 2, \cdots \\ q = 1, 2, 3, \cdots \end{array} \tag{16}$$

represents the "normalized" scattering-wave vector, with coefficients set equal to unity. The coefficient vectors $\left\{ A_S^k \right\}$ for the scattering potential and $\left\{ \mathcal{R}_j^{\ell k} \right\}$ for the radiation potential, differ depending on each problem. These can be determined by Kagemoto & Yue's wave interaction theory [3], but it should be noted that $\left\{ \mathcal{R}_j^{\ell k} \right\}$ includes two components: 1) the radiation wave due to forced oscillation in the j-th mode of floating body ℓ alone, and 2) the scattering wave obtained by regarding the reflected waves from the other bodies ($k \neq \ell$) as incident waves to floating body ℓ. Thus, it should be written as

$$\left\{ \mathcal{R}_j^{\ell k} \right\} = \left\{ R_j^\ell \right\} \delta_{k\ell} + \left\{ A_j^{\ell k} \right\} \tag{17}$$

where $\left\{ R_j^\ell \right\}$ and $\left\{ A_j^{\ell k} \right\}$ denote the coefficient vectors of the first and second components, respectively.

In the far field ($r \gg 1$) at a distance from all floating bodies, evanescent waves decay, and the outward propagating wave component represented by $H_p^{(2)}(k_0 r_k)$ can be expressed with global coordinates using the following addition theorem valid for $r > L_{ok}$ (see Fig. 2):

$$H_p^{(2)}(k_0 r_k)\, e^{-ip\theta_k} = \sum_{n=-\infty}^{\infty} J_{n-p}(k_0 L_{ok})\, e^{i(n-p)\alpha_{ok}} \left\{ H_n^{(2)}(k_0 r)\, e^{-in\theta} \right\} \tag{18}$$

Collecting (15)–(18) and utilizing the far-field asymptotic expression of $H_n^{(2)}(k_0 r)$, we can obtain and write the disturbance potential at a far field in the form

$$\phi_B(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \mathcal{A}_n \left\{ Z_0(z) H_n^{(2)}(k_0 r)\, e^{-in\theta} \right\} \simeq \sqrt{\frac{2}{\pi k_0 r}} \left\{ \sum_{n=-\infty}^{\infty} (i)^n \mathcal{A}_n\, e^{-in\theta} \right\} Z_0(z)\, e^{-ik_0 r + i\pi/4} \tag{19}$$

where

$$\mathcal{A}_n = \mathcal{A}_{Sn} - K \sum_{\ell=1}^{N_B} \sum_{j=1}^{6} X_j^\ell \mathcal{A}_{jn}^\ell, \quad \left\{ \begin{array}{c} \mathcal{A}_{Sn} \\ \mathcal{A}_{jn}^\ell \end{array} \right\} = \sum_{k=1}^{N_B} \sum_{p=-\infty}^{\infty} J_{n-p}(k_0 L_{ok})\, e^{i(n-p)\alpha_{ok}} \left\{ \begin{array}{c} A_{Sp}^k \\ \mathcal{R}_{jp}^{\ell k} \end{array} \right\} \tag{20}$$

Note that the p-th elements of the coefficient vectors $\left\{ A_S^k \right\}$ and $\left\{ \mathcal{R}_j^{\ell k} \right\}$ are expressed as A_{Sp}^k and $\mathcal{R}_{jp}^{\ell k}$, respectively, and $\mathcal{A}_n$ in (20) is the n-th element of superimposed coefficient vector of the disturbance potential.

Comparison of this result for $\phi_B(\boldsymbol{x})$ with the corresponding expression in BEM to be obtained from (7) and (12)–(13) reveals that the following relationships hold:

$$(i)^n \mathcal{A}_n = \left(\frac{i}{2} C_0 \right) c_n, \quad \left\{ \begin{array}{c} \mathcal{A}_{Sn} \\ \mathcal{A}_{jn}^\ell \end{array} \right\} = \frac{i}{2} C_0 \left\{ \begin{array}{c} c_{Sn} \\ c_{jn}^\ell \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} A_{Sp}^k \\ \mathcal{R}_{jp}^{\ell k} \end{array} \right\} = \frac{i}{2} C_0 \left\{ \begin{array}{c} c_{Sp}^k \\ c_{jp}^{\ell k} \end{array} \right\} \tag{21}$$

From these results, we can see that the coefficient-vector element $\mathcal{A}_n$ in the eigenfunction-expansion method has a one-to-one relationship with the complex Fourier-series coefficient c_n when the Kochin function is expressed as the complex Fourier series.

3 Hydrodynamic Relations for Multiple Floating Bodies

Some important hydrodynamic relationships can be derived using Green's formula. Considering the control surface encompassing all the floating bodies, the following equation holds:

$$\sum_{m=1}^{N_B} \iint_{S_m} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS = - \iint_{S_\infty} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS \tag{22}$$

The positive normal direction is taken to be inward of the fluid domain. The preconditions for two different velocity potentials φ and ψ are that both satisfy the same boundary conditions on the free surface and the water bottom, but not necessarily the same on the floating-body surface (S_m) and the control surface (S_∞).

3.1 Energy-conservation relation in the radiation problem

First, considering $\varphi = \phi_j^\ell$ and $\psi = \phi_i^k$ in the radiation problem, we can prove that the identity of (22) yields the symmetry relation in the added mass and damping coefficient, expressed in the form of $A_{ij}^{k\ell} = A_{ji}^{\ell k}$ and $B_{ij}^{k\ell} = B_{ji}^{\ell k}$. This symmetry relation means that the interchange rule holds not only for mode numbers i and j, but also for floating-body numbers k and ℓ.

Next, consider $\varphi = \phi_j^\ell$ and $\psi = \phi_i^{k*}$ (complex conjugate of ϕ_i^k). Taking account of the symmetry relation and the far-field asymptotic form of ϕ_j^ℓ given in (7), the identity of (22) provides the following relationship:

$$B_{ij}^{k\ell} = \frac{C_0}{4\pi}\int_0^{2\pi} H_j^\ell(k_0,\theta)\, H_i^{k*}(k_0,\theta)\, d\theta = \frac{C_0}{2}\sum_{n=-\infty}^{\infty} c_{jn}^\ell\, c_{in}^{k*} = \frac{2}{C_0}\sum_{n=-\infty}^{\infty} \mathcal{A}_{jn}^\ell\, \mathcal{A}_{in}^{k*} \tag{23}$$

Here, we have used (12) for the Kochin function and (21) for the eigenfunction-expansion coefficient.

3.2 Haskind-Newman relation for wave-exciting force

Let us consider the combination of $\varphi = \phi_D$ and $\psi = \phi_j^\ell$. By taking account of the body boundary condition $[\,H\,]$ in (5), the left-hand side of (22) can be found to be the wave-exciting force E_j^ℓ. For the right-hand side, we note that ϕ_S in ϕ_D and ϕ_j^ℓ satisfy the same radiation condition on S_∞, and hence only the combination of ϕ_I and ϕ_j^ℓ may be considered. For these velocity potentials, we adopt the expressions in the eigenfunction expansion; that is, (3) and (19) on $z = 0$. Then, using the orthogonal relation in trigonometric functions and the Wronskian for Bessel functions yields the following relation known as Haskind-Newman relation:

$$E_j^\ell = \iint_{S_\ell} \phi_D\, n_j^\ell\, dS = -i\frac{2}{C_0}\sum_{n=-\infty}^{\infty}(-i)^n\, e^{-in\beta}\,\mathcal{A}_{jn}^\ell = \sum_{n=-\infty}^{\infty}(i)^n\, c_{jn}^\ell\, e^{-in(\beta+\pi)} = H_j^\ell(k_0,\beta+\pi) \tag{24}$$

We have used the relation in (21) between $\mathcal{A}_{jn}^\ell$ and c_{jn}^ℓ, and also (12) for the radiation Kochin function.

As the control surface S_∞ in Green's formula, we may consider a vertical circular cylinder encompassing only the floating body ℓ under consideration. In this case, however, the effects of evanescent waves may not be ignored. In order to include these effects, the analysis can be made with vector and matrix notations. We note that the incident-wave potential on this control surface includes the waves reflected from all the other bodies and incoming to floating body ℓ, in addition to the original incident wave from the far outside.

Therefore, the total incident wave potential on floating body ℓ can be given as follows:

$$\phi_I^\ell = \phi_I + \sum_{k=1,k\neq\ell}^{N_B}\{A_S^k\}^T\{\psi_S^k\} = \Big(\{a^\ell\}^T + \sum_{k=1,k\neq\ell}^{N_B}\{A_S^k\}^T\big[T_{k\ell}\big]\Big)\{\psi_I^\ell\} \equiv \{\mathcal{A}_I^\ell\}^T\{\psi_I^\ell\} \tag{25}$$

where

$$\{a^\ell\}^T = \Big\{(-i)^m\, e^{im\beta}\, E_\ell(\beta), \quad 0\Big\}, \quad E_\ell(\beta) = e^{-ik_0(x_{o\ell}\cos\beta + y_{o\ell}\sin\beta)} \tag{26}$$

$$\Big\{\psi_I^\ell\Big\} = \left\{\begin{array}{c} Z_0(z)\, J_m(k_0 r_\ell)\, e^{-im\theta_\ell} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Z_n(z)\, I_m(k_n r_\ell)\, e^{-im\theta_\ell}\end{array}\right\} \quad \begin{array}{l} m = 0, \pm 1, \pm 2, \cdots \\ n = 1, 2, 3, \cdots \end{array} \tag{27}$$

and $\big[T_{k\ell}\big]$ is the coordinate transformation matrix, composed of the addition theorem of Bessel functions and the position vector of the origin of floating body ℓ viewed from the origin of floating body k.

On the other hand, the radiation potential ϕ_j^ℓ on this new control surface should be the single-body radiation potential, thus we can use only the first term on the right-hand side of (17). Substituting (25) and the eigenfunction-expansion equation for ϕ_j^ℓ, and also utilizing orthogonality and Wronskian relations including local evanescent-wave terms, we can obtain the following result:

$$E_j^\ell = \{\mathcal{A}_I^\ell\}^T\{e_j^\ell\}, \quad \{e_j^\ell\}^T = \Big\{-i\frac{2}{C_0}(-i)^m R_{j,-m}^\ell, \ \frac{\pi}{C_n} R_{j,-mn}^\ell\Big\} \tag{28}$$

It can be seen that (28) is in the same form as the calculation formula for the wave-exciting force in Kagemoto & Yue's wave interaction theory, in which however $\{e_j^\ell\}$ will be computed directly from the normalized diffraction potential vector $\{\psi_D^\ell\}$ determined by solving an integral equation, with $\{\psi_I^\ell\}$ used as the forcing term.

References

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