On the linearization of the seakeeping problem at forward speed

Quentin Derbanne
Bureau Veritas Marine & Offshore, Nantes
quentin.derbanne@bureauveritas.com

1 INTRODUCTION

1.1 General problem

The general problem for the potential Φ describing the relative flow in the translating reference system at speed U, is written in a dimensionless way, where all quantities are made dimensionless using the ship length L, the gravity g and the water density ρ. Ω is the fluid domain, bounded by the body surface B and the free surface η.

$$\begin{gathered} \text{in } \Omega: \quad \Delta\Phi = 0 \qquad \Phi \underset{\infty}{\rightarrow} -Fx + \Phi_I \qquad P = P_a - z - \Phi_t + \frac{1}{2}(F^2 - \nabla\Phi^2) \\ \Phi_z + \Phi_{tt} + 2\nabla\Phi\nabla\Phi_t + \frac{1}{2}\nabla\Phi\nabla(\nabla\Phi^2) = 0 \quad \text{on } z = \eta = -\Phi_t + \frac{1}{2}(F^2 - \nabla\Phi^2) \\ \Phi_n = \left(\vec{V} + \vec{\omega} \times \overrightarrow{GM}\right) \cdot \vec{\mathrm{n}} \quad \text{on } B \end{gathered} \tag{1}$$

$\vec{V}\sqrt{gL}$ and $\vec{\omega}\sqrt{g/L}$ are the translational and rotational velocities of the body, $F = U/\sqrt{gL}$ is the Froude number, $P_a \rho g L$ is the atmospheric pressure and $\Phi_I L\sqrt{gL}$ is the incident potential.

1.2 Problem decomposition

The potential is usually decomposed into a steady base flow ϕ, a steady perturbation $\bar{\varphi}$ and an unsteady perturbation φ. Both the steady and the unsteady problems are fully nonlinear because of nonlinear terms appearing in the equations and because some boundary conditions should be satisfied on unknown locations (deformed free surface position or instantaneous body position). Linearization is a practical solution to try to solve these problems. It relies on three successive steps: removal of high order terms in the equations (the perturbation potentials $\bar{\varphi}$ and φ are supposed to be "small" compared to the base flow ϕ), Taylor series expansion of the boundary conditions, and truncation of the equations by removing "smaller" terms.

1.3 Points to be clarified

These linearization steps raise several questions. Is there a unique choice for the base flow? What is the meaning of "small"? How to justify that some terms are "smaller" than others? Where to apply the Taylor expansion? How to truncate it?

Many authors have tackled the steady flow problem, leading to very different linearized free surface conditions. Dawson [1] simply removes the high order terms and applies the free surface condition on z=0, without any justification. Eggers [2] "axiomatically" assumes that the perturbation potential is $\mathcal{O}(F^3)$. Sclavounos [3] assumes "the basis wave elevation is a sufficiently small quantity", knowing this is not true "near the ship waterline and in particular near the ship bow". The seakeeping problem at forward speed cannot be considered independently from the steady problem. Hence, for consistency, methods and assumptions used to solve the steady problem should also be used to solve the unsteady problem.

2 DEFINITIONS

The term "order" is usually employed to name two different things, and it is of crucial importance to have a common understanding about this.

2.1 Order of magnitude

An order of magnitude is a logarithmic measure that classifies the scale of a quantity based on powers of ten. A proper definition is:

$$ord(a) = 10^n \quad \Leftrightarrow \quad 1/\sqrt{10} \leq |a|/10^n < \sqrt{10} \qquad n \in \mathbb{N} \tag{2}$$

In most cases, the order of magnitude of a quantity is unknown until the problem has been solved and this quantity computed. Hence it is not possible to use the order of magnitude to simplify the equations.

2.2 Order of approximation

Saying that a function $g(x)$ is $\mathcal{O}(x^n)$ is saying that $g(x)$ behaves like ax^n when x tends to zero, with a being a nonzero constant. A proper definition is:

$$g(x) = \mathcal{O}(x^n) \quad \Leftrightarrow \quad \lim_{x\to 0}\left(\frac{g}{x^n}\right) = a \ \text{ with } \ 0 < |a| < \infty \tag{3}$$

If a function $g(x)$ is of order $\mathcal{O}(x^n)$ and a function $f(x)$ is of order $\mathcal{O}(x^{n+1})$, then $f(x)$ can be considered "smaller" than $g(x)$. It means that, whatever ϵ, there exists a value X for which $|f(x)| < \epsilon|g(x)|$ if $|x| < X$. This can be easily proven using (3). However, it doesn't mean anything for the order of magnitude of X: X may be much greater or much smaller than 1. Likewise, it doesn't mean that the order of magnitude of $g(x)$ is equal to the order of magnitude of x^n, because we don't know the order of magnitude of a.

2.3 Taylor series

The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. A function $g(x)$ is called entire if it is equal to its Taylor series for all x. The partial sum formed by the n first terms of a Taylor series is a polynomial of degree n that is called the n$^{\text{th}}$ Taylor polynomial of the function. The remaining terms are $\mathcal{O}((x-x_0)^{n+1})$.

$$g(x) = \sum_{i=0}^{n} \frac{1}{i!}\frac{\partial^i g}{\partial x^i}(x_0)\,(x-x_0)^i + \mathcal{O}((x-x_0)^{n+1}) \tag{4}$$

Taylor polynomials can be used to simplify boundary conditions and to apply them on known surfaces. The order of approximation of the missing terms is known, meaning that, for small enough Froude numbers or wave amplitude, the order of magnitude of these missing terms could be as small as we want.

3 PROBLEM LINEARIZATION

Because, for an efficient numerical resolution, we want to have two independent linear problems for $\bar{\varphi}$ and φ, we need to apply the following steps.

3.1 Decomposition of the potential

The total potential is decomposed in $\Phi = \phi + \bar{\varphi} + \varphi$. The base flow $\nabla\phi$ is necessary $\mathcal{O}(F)$ as it must be equal to $-Fx$ at infinity. The steady perturbation $\nabla\bar{\varphi}$ is supposed to be $\mathcal{O}(F^n)$ with n being unknown for now. The unsteady perturbation $\nabla\varphi$ is of order $\mathcal{O}(\varepsilon)$. The choice of the base flow ϕ is not unique, but the resulting systems of equations for $\bar{\varphi}$ and φ strongly depend on this choice.

3.2 Removal of high order terms

To obtain two independent linear systems of equations for in $\bar{\varphi}$ and φ, we are forced to remove all the quadratic and cubic terms in $\bar{\varphi}$ and φ from the different equations. This imposes the truncation orders of the different equations. The free surface boundary condition is truncated at $\mathcal{O}(F^{2n+1}) + \mathcal{O}(F^n\varepsilon) + \mathcal{O}(\varepsilon^2)$, and the pressure and free surface elevation is truncated at $\mathcal{O}(F^{2n}) + \mathcal{O}(F^n\varepsilon) + \mathcal{O}(\varepsilon^2)$.

3.3 Taylor development of the boundary conditions

All free surface and body boundary conditions are developed in Taylor series and truncated at the orders defined in 3.2. The different terms in the free surface elevation are $\mathcal{O}(F^2)$, $\mathcal{O}(F^{n+1})$, $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon F)$. They cannot be considered as "small" and should be properly considered in the Taylor development. In particular, the term $\frac{1}{2}(F^2 - \nabla\phi^2)$ is $\mathcal{O}(F^2)$ and is noted F^2h in the following equations. This result in two independent linear systems of equations for $\bar{\varphi}$ and φ that are applied on $z = 0$ and the steady body position B_0. The terms coming from the Taylor development are indicated in red in equation (5). We have achieved our goal: we have two independent linear systems of equations for $\bar{\varphi}$ and φ.

$$\boxed{\eta = \left(\begin{array}{c}\underbrace{F^2h}_{\mathcal{O}(F^2)} - \underbrace{F^2h\nabla\phi\nabla\phi_z}_{\mathcal{O}(F^4)} - \underbrace{\nabla\phi\nabla\bar{\varphi}}_{\mathcal{O}(F^{n+1})} - \underbrace{\varphi_t}_{\mathcal{O}(\varepsilon)} - \underbrace{\nabla\phi\nabla\varphi}_{\mathcal{O}(\varepsilon F)} + \underbrace{2\varphi_t\nabla\phi\nabla\phi_z - F^2h\varphi_{tz}}_{\mathcal{O}(\varepsilon F^2)} \\ +\mathcal{O}(F^6) + \mathcal{O}(F^{n+3}) + \mathcal{O}(F^{2n}) + \mathcal{O}(\varepsilon F^3) + \mathcal{O}(\varepsilon F^n) + \mathcal{O}(\varepsilon^2)\end{array}\right) \quad \text{on } z = 0} \tag{5}$$

$$
\left(\begin{aligned}
&\underbrace{\bar{\varphi}_z}_{O(F^n)} + \underbrace{\nabla\phi\nabla(\nabla\phi\nabla\bar{\varphi}) + \tfrac{1}{2}\nabla\bar{\varphi}\nabla(\nabla\phi^2) + F^2 h\bar{\varphi}_{zz} - \phi_{zz}\nabla\phi\nabla\bar{\varphi}}_{O(F^{n+2})} \\
&= -\underbrace{\phi_z}_{O(F)} - \underbrace{\tfrac{1}{2}\nabla\phi\nabla(\nabla\phi^2) + F^2 h\phi_{zz}}_{O(F^3)} + \underbrace{F^2 h\phi_{zz}\nabla\phi\nabla\phi_z + \tfrac{1}{2}F^4 h^2\phi_{zzz}}_{O(F^5)} \\
&\quad -\underbrace{\tfrac{1}{2}F^2 h\nabla\phi_z\nabla(\nabla\phi^2) + F^2 h\nabla\phi\nabla(\nabla\phi\nabla\phi_z)}_{O(F^5)} + O(F^{2n+1}) + O(F^{3n})
\end{aligned}\right) \quad \text{on } z = 0
$$

$$
\underbrace{\bar{\varphi}_n}_{O(F^n)} = -\underbrace{\phi_n}_{O(F)} \quad \text{on } B_0
$$

$$
\left(\begin{aligned}
&\underbrace{\varphi_z + \varphi_{tt}}_{O(\varepsilon)} + \underbrace{2\nabla\phi\nabla\varphi_t - \phi_{zz}\varphi_t}_{O(\varepsilon F)} + \underbrace{\nabla\phi\nabla(\nabla\phi\nabla\varphi) + \tfrac{1}{2}\nabla\varphi\nabla(\nabla\phi^2)}_{O(\varepsilon F^2)} \\
&\quad + \underbrace{F^2 h\varphi_{zz} + F^2 h\varphi_{ttz} - \phi_{zz}\nabla\phi\nabla\varphi}_{O(\varepsilon F^2)} = O(\varepsilon F^n) + O(\varepsilon^2)
\end{aligned}\right) \quad \text{on } z = 0
$$

$$
\underbrace{\varphi_n}_{O(\varepsilon)} = \left(\begin{aligned}
&\underbrace{\left(\vec{V} + \vec{\omega}\times\overrightarrow{G_0M_0}\right)\cdot\overrightarrow{n_0}}_{O(\varepsilon)} + \underbrace{\vec{\theta}\cdot\nabla\phi\times\overrightarrow{n_0} - \phi_{nn}\left(\overrightarrow{G_0G}\cdot\overrightarrow{n_0} + \vec{\theta}\cdot\overrightarrow{G_0M_0}\times\overrightarrow{n_0}\right)}_{O(\varepsilon F)} \\
&\quad + O(\varepsilon F^n) + O(\varepsilon F^3) + O(\varepsilon^2)
\end{aligned}\right) \quad \text{on } B_0
$$

4 CHOICE OF THE BASE FLOW

The choice of the base flow is free, but only a clever choice will give useful equations.

4.1 Uniform flow

The simplest choice for the base flow is $\phi = -Fx$. In this case we have $h = 0$, and the equations become very simple. The right side of the body boundary condition is $O(F)$. Hence the steady perturbation must be $O(F)$. It means that $n = 1$. Therefore, for consistency, all terms $O(F^2)$ and (εF) should be removed from the free surface elevation, and all terms $O(F^3)$ and $O(\varepsilon F)$ should be removed from the boundary conditions:

$$
\bar{\eta} = \underbrace{\cancel{F\bar{\varphi}_x}}_{\cancel{O(F^2)}} - \underbrace{\varphi_t}_{O(\varepsilon)} + \underbrace{\cancel{F\varphi_x}}_{\cancel{O(\varepsilon F)}} + O(F^2) + O(\varepsilon F) + O(\varepsilon^2) \quad \text{on } z = 0
$$

$$
\underbrace{\bar{\varphi}_z}_{O(F)} + \underbrace{\cancel{F^2\bar{\varphi}_{xx}}}_{\cancel{O(F^3)}} = O(F^3) \quad \text{on } z = 0 \qquad \underbrace{\bar{\varphi}_n}_{O(F)} = \underbrace{Fn_x}_{O(F)} \quad \text{on } B_0 \tag{6}
$$

$$
\underbrace{\varphi_z + \varphi_{tt}}_{O(\varepsilon)} - \underbrace{\cancel{2F\varphi_{tx}}}_{\cancel{O(\varepsilon F)}} + \underbrace{\cancel{F^2\varphi_{xx}}}_{\cancel{O(\varepsilon F^2)}} = O(\varepsilon F) + O(\varepsilon^2) \quad \text{on } z = 0
$$

$$
\underbrace{\varphi_n}_{O(\varepsilon)} = \underbrace{\left(\vec{V} + \vec{\omega}\times\overrightarrow{G_0M_0}\right)\cdot\overrightarrow{n_0}}_{O(\varepsilon)} + \underbrace{\cancel{F\theta_y n_z - F\theta_z n_y}}_{\cancel{O(\varepsilon F)}} + O(\varepsilon F) + O(\varepsilon^2) \quad \text{on } B_0
$$

The remaining terms form the double-body flow problem for $\bar{\varphi}$ and the encounter frequency problem for φ. Note that the usual m_j terms in the body condition have been removed as they are $O(\varepsilon F)$. The choice of the uniform flow for the base flow naturally leads to a very simplified problem for the unsteady potential.

4.2 Double-body flow

The other possible choice for the base flow is $\phi = F(\chi - x)$, where χ satisfies $\chi_z = 0$ on $z = 0$ and $\chi_n = n_x$ on the body. In this case we have $h \neq 0$. The right side of the body boundary condition is now 0. The leading order in the free surface equation is $O(F^3)$, forcing the steady perturbation to be $O(F^3)$. It means that $n = 3$. Therefore, the truncation orders are now $O(F^6) + O(\varepsilon F^3) + O(\varepsilon^2)$ for the free surface elevation and $O(F^7) + O(\varepsilon F^3) + O(\varepsilon^2)$ for the boundary conditions.

$$
\eta = \underbrace{F^2 h}_{O(F^2)} - \underbrace{\nabla\phi\nabla\bar{\varphi}}_{O(F^4)} - \underbrace{\varphi_t}_{O(\varepsilon)} - \underbrace{\nabla\phi\nabla\varphi}_{O(\varepsilon F)} - \underbrace{F^2 h\varphi_{tz}}_{O(\varepsilon F^2)} + O(F^6) + O(\varepsilon F^3) + O(\varepsilon^2) \quad \text{on } z = 0 \tag{7}
$$

$$\left(\begin{array}{c}\underbrace{\bar{\varphi}_z}_{\mathcal{O}(F^3)} + \underbrace{\nabla\phi\nabla(\nabla\phi\nabla\bar{\varphi}) + \tfrac{1}{2}\nabla\bar{\varphi}\nabla(\nabla\phi^2) + F^2h\bar{\varphi}_{zz}}_{\mathcal{O}(F^5)} \\ -\underbrace{\nabla\phi_{zz}\nabla\phi\nabla\bar{\varphi}}_{\mathcal{O}(F^5)} = -\underbrace{\tfrac{1}{2}\nabla\phi\nabla(\nabla\phi^2) + F^2h\phi_{zz}}_{\mathcal{O}(F^3)} + \mathcal{O}(F^7)\end{array}\right) \text{on } z=0 \qquad \underbrace{\bar{\varphi}_n}_{\mathcal{O}(F^n)} = -\underbrace{\phi_n}_{\mathcal{O}(F)} \text{ on } B_0$$

$$\left(\begin{array}{c}\underbrace{\varphi_z + \varphi_{tt}}_{\mathcal{O}(\varepsilon)} + \underbrace{2\nabla\phi\nabla\varphi_t - \phi_{zz}\varphi_t}_{\mathcal{O}(\varepsilon F)} + \underbrace{\nabla\phi\nabla(\nabla\phi\nabla\varphi) + \tfrac{1}{2}\nabla\varphi\nabla(\nabla\phi^2)}_{\mathcal{O}(\varepsilon F^2)} \\ + \underbrace{F^2h\varphi_{zz} + F^2h\varphi_{ttz} - \phi_{zz}\nabla\phi\nabla\varphi}_{\mathcal{O}(\varepsilon F^2)} = \mathcal{O}(\varepsilon F^3) + \mathcal{O}(\varepsilon^2)\end{array}\right) \text{on } z=0$$

$$\underbrace{\varphi_n}_{\mathcal{O}(\varepsilon)} = \left(\begin{array}{c}\underbrace{(\vec{V} + \vec{\omega}\times\overrightarrow{G_0M_0})\cdot\overrightarrow{n_0}}_{\mathcal{O}(\varepsilon)} + \underbrace{\vec{\theta}\cdot\nabla\phi\times\overrightarrow{n_0} - \phi_{nn}(\overrightarrow{G_0G}\cdot\overrightarrow{n_0} + \vec{\theta}\cdot\overrightarrow{G_0M_0}\times\overrightarrow{n_0})}_{\mathcal{O}(\varepsilon F)} \\ +\mathcal{O}(\varepsilon F^3) + \mathcal{O}(\varepsilon^2)\end{array}\right) \text{on } B_0$$

5 DISCUSSION

Many authors ([3], [4]) are considering the entire free surface elevation as “small” and are applying the free surface boundary condition directly on $z = 0$ without any correction terms. We have shown here that the leading term of the free surface elevation, F^2h, is $\mathcal{O}(F^2)$ and should be properly considered in the Taylor development. All the equations presented here are only valid at small Froude numbers, and the resulting equations for φ entirely depend on the choice of the base flow ϕ.

The choice of the uniform flow leads to equation (6), which is truncated at $\mathcal{O}(\varepsilon F)$ and describes the encounter frequency approximation (without the usual m_j terms). The Neumann-Kelvin approach appears to be inconsistent as it considers some terms $\mathcal{O}(\varepsilon F)$ and $\mathcal{O}(\varepsilon F^2)$ but neglects other terms of the same order. The only choice that leads to a “small” steady perturbation $\mathcal{O}(F^3)$ is the double-body flow as it satisfies $\phi_z = 0$ on $z = 0$ and $\phi_n = 0$ on the body. This gives equation (7), which is truncated at $\mathcal{O}(\varepsilon F^3)$. If we remove all the terms $\mathcal{O}(\varepsilon F^2)$ from equation (7) we obtain the small current approximation. Any other choice for ϕ will make the steady perturbation $\mathcal{O}(F)$ and will force to truncate the unsteady problem at $\mathcal{O}(\varepsilon F)$.

It is true that equation (7) may cause some numerical problems: The terms $\varphi_z + F^2h\varphi_{ttz}$, once transformed in the frequency domain become $\varphi_z(1 - \tau^2h)$, with $\tau = \omega U/g$. The term in the brackets might become zero or negative, which will produce some instabilities in the boundary integral equations. It doesn’t mean that this corrective term should not exist. On the opposite it means that the wave elevation F^2h is too big compared to the characteristic scale of vertical variations of φ_{tt}. In other words, for high values of τ, and only at some locations on the free surface where h is big (at bow and stern, close to the vessel), the Taylor development on $z = 0$ is not licit.

Of course, one solution would be to linearize the free surface equation on $z = F^2h$ instead of $z = 0$. This would automatically remove all the terms proportional to h in the equations. However, this would force the solver to use a Rankine type method, including the meshing of the free surface. The Kelvin type Green function could not be used in this case.

REFERENCES

[1] Dawson, C.W. 1977. "*A Practical Computer Method for Solving Ship-Wave Problems*", 2nd lnt. Conf. Numerical Ship Hydrodynamics, Berkeley, USA.
[2] Eggers, K. 1981. "*Non-Kelvin dispersive waves around non-slender ships* ", Schiffstechnik, Bd. 28.
[3] Sclavounos, P.D. and Ohkusu, M. 1996. “*Chapter 4. Computation of wave ship interactions*”, in Advances in Fluid Mechanics.
[4] Chen, X., Nguyen, M., Ten, I., Ouled Housseine, C., Choi, Y., Diebold, L., Malenica, S., De-Hauteclocque, G., and Derbanne, Q. 2022. *“New seakeeping computations based on potential flows linearised over the ship-shaped stream”*, 15th Int. Symp. on Practical Design of Ships and Other Floating Structures.