

Ice-Shelf Break-Up Simulation in a Bounded Domain

Faraj Alshahrani^{a,b}, Michael H. Meylan^a, Ben Wilks^{a,c}

^a School of Computer and Information Sciences, University of Newcastle, Callaghan, Australia

^b Department of Mathematics, Prince Sattam bin Abdulaziz University, Al-Kharj, Saudi Arabia

^c UniSA STEM, The University of South Australia, Mawson Lakes, Australia

Email: fs.alshahrani@psau.edu.sa

1 HIGHLIGHTS

- A Kirchhoff-Love plate theory and shallow water theory are combined and solved using a Fourier series method.
- Ice shelf breakup is simulated, with fracture events occurring at discrete time points based on a local strain criterion.

2 INTRODUCTION

The rise in sea level is a major challenge in climate change research, and the breaking of ice shelves is crucial to understanding this increase in sea level (Sergienko, 2010; Alshahrani et al., 2024). Understanding the breakup process improves sea level forecasts and informs mitigation and adaptation (Alshahrani et al., 2025). We develop a mathematical model to simulate ice shelf dynamics and solve it using a Fourier series method. Breakup in response to incident wave forcing is then simulated using a local strain breakup criterion.

3 MATHEMATICAL MODEL

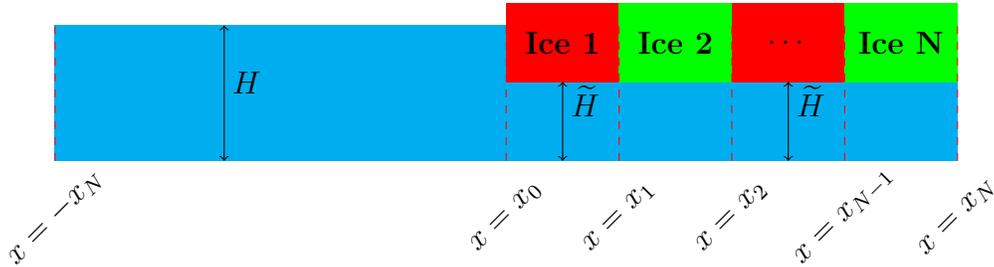


Figure 1: Schematic of a multi-segment ice configuration.

We consider a floating ice shelf of thickness h over water of total depth H , as illustrated in Figure 1. The ice shelf has flexural rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$, where $E = 11$ GPa is the Young's modulus and $\nu = 0.33$ is the Poisson ratio. The water and ice densities are taken as $\rho_w = 1025 \text{ kg m}^{-3}$ and $\rho_I = 922.5 \text{ kg m}^{-3}$, respectively, so that the vertical gap between the submerged underside of the ice shelf and the seabed is $\tilde{H} = H - \left(\frac{\rho_I}{\rho_w}\right)h$. The acceleration due to gravity is $g = 9.81 \text{ m s}^{-2}$ (Meylan, 2002; Sergienko, 2010). The geometry of the problem is shown in Figure 1. Kirchhoff–Love thin plate theory and shallow water theory are applied to model the problem, which combine to give the following sixth order equation for the velocity potential of the fluid:

$$-\omega^2\phi(x) = \begin{cases} Hg\partial_x^2\phi^{(0)}(x), & x \in (-x_N, x_0), \\ \frac{\tilde{H}}{\rho_w}(D\partial_x^6\phi^{(i)}(x) + (\rho_w g - \omega^2\rho_I h)\partial_x^2\phi^{(i)}(x)), & x \in (x_{i-1}, x_i), \end{cases} \quad (1)$$

plus boundary conditions at $x = \pm L$ and matching conditions at the water/ice interface. In particular, at the left boundary $x = -x_N$ we enforce no flux. At the first ice edge $x = x_0$ we require continuity of Φ and its flux, and impose the free-edge conditions on the ice shelf. At each internal interface $x = x_i$, $i = 1, \dots, N - 1$, Φ and its flux remain continuous, and the free-edge conditions apply on both sides. At the rightmost end $x = x_N$ we allow two terminal conditions for the ice shelf: clamped or free-edge.

3 EIGENMODES

The general solution to (1) for $\omega > 0$ is

$$\phi(x, \omega) = \begin{cases} R_1 e^{ikx} + R_2 e^{-ikx}, & x \in (-x_N, x_0), \\ \sum_{n=1}^3 \alpha_n^{(i)} e^{r_n^{(i)}(x-x_{i-1})} + \beta_n^{(i)} e^{-r_n^{(i)}(x-x_i)}, & x \in (x_{i-1}, x_i). \end{cases} \quad (2a)$$

The corresponding free surface/plate displacement is

$$w(x, \omega) = \begin{cases} \frac{-Hk^2}{i\omega} (R_1 e^{ikx} + R_2 e^{-ikx}), & x \in (-x_N, x_0), \\ \sum_{n=1}^3 \frac{\tilde{H}(r_n^{(i)})^2}{i\omega} \left(\alpha_n^{(i)} e^{r_n^{(i)}(x-x_{i-1})} + \beta_n^{(i)} e^{-r_n^{(i)}(x-x_i)} \right) & x \in (x_{i-1}, x_i). \end{cases} \quad (2b)$$

Applying the boundary conditions lead to a homogeneous linear algebraic system for the coefficients in the eigenfunction expansion of Φ , written compactly as

$$M(\omega) X = 0, \quad (3)$$

where $M(\omega)$ is a $(6N + 2) \times (6N + 2)$ matrix depending on the angular frequency ω , and

$$X = \left(R_1, R_2, \{ \alpha_n^{(i)} \}_{i=1, \dots, N}^{n=1,2,3}, \{ \beta_n^{(i)} \}_{i=1, \dots, N}^{n=1,2,3} \right)^T \in \mathbb{C}^{6N+2}.$$

is the vector of unknown modal coefficients. The system also supports a static mode which needs to be addressed using a separate procedure. The static mode is analogous to the problem of ice shelf bending due to tidal loading (Vaughan, 1995). The vertical displacement of the static mode is

$$w_0(x) = \begin{cases} A, & x \in (-x_N, x_0), \\ A + \sum_{j=1}^4 C_j^{(i)} f_j^{(i)}(x), & x \in (x_{i-1}, x_i), \end{cases} \quad (4)$$

where $s = e^{i\pi/4} \left(\frac{\rho_w g}{D} \right)^{1/4}$ and the basis functions in (4) are

$$f_j^{(i)}(x) = [\cos(s(x - x_i)), \sin(s(x - x_i)), e^{s(x-x_i)}, e^{-sx}]. \quad (5)$$

After applying the boundary and interface matching conditions, we obtain a system of equations for the coefficients $C_j^{(i)}$ of the form

$$\mathbf{M} \mathbf{C} = \mathbf{b}, \quad (6)$$

with $\mathbf{C} = [C_{11}, C_{12}, C_{13}, C_{14}, \dots, C_{N4}]^T$, which we solve to obtain the static bending profile.

4 TIME DOMAIN SOLUTION

We conduct simulations with a truncated series representation using N_{modes} . The associated vertical displacement is

$$W(x, t) = (C_0 + D_0 t)w_0(x) + \sum_{n=1}^{N_{\text{modes}}} \left[C_n \cos(\omega_n t) + D_n \frac{\sin(\omega_n t)}{\omega_n} \right] w_n(x). \quad (7)$$

The coefficients C_n and D_n are determined using initial conditions of the form

$$f(x) = W(x, 0) = \sum_{n=0}^{\infty} C_n w_n(x), \quad \text{and} \quad g(x) = \partial_t W(x, 0) = \sum_{n=0}^{\infty} D_n w_n(x). \quad (8)$$

To determine the coefficients C_n and D_n , we require the appropriate inner product for our problem, which is derived from the potential energy as

$$\langle w_m, w_n \rangle_{\mathcal{V}} = \rho_w g \int_{-L}^L w_m \overline{w_n} dx + D \int_0^L w_m \frac{\partial^4 \overline{w_n}}{\partial x^4} dx. \quad (9)$$

The coefficients are recovered using quadrature

$$\begin{Bmatrix} C_n \\ D_n \end{Bmatrix} \approx \rho_w g \sum_{i=1}^N \begin{Bmatrix} f(x_i) \\ g(x_i) \end{Bmatrix} w_n(x_i) \Delta x + D \sum_{i=\frac{N+1}{2}}^N \begin{Bmatrix} f(x_i) \\ g(x_i) \end{Bmatrix} \frac{\partial^4 w_n(x_i)}{\partial x^4} \Delta x. \quad (10)$$

5 RESULTS

The simulation starts by initializing $W(x, t)$ with (7) and the initial conditions. During evolution, the curvature $W_{xx}(x, t)$ is monitored for $x \geq 0$. A crack forms when

$$\max_{x \geq 0} |W_{xx}(x, t)| \geq a, \quad (11)$$

for some threshold $a > 0$. Let $t = t_k$ be the first time this occurs. The crack location is

$$x_k = \arg \max_{x \geq 0} |W_{xx}(x, t_k)|, \quad (12)$$

and the crack time is t_k . When a crack appears at x_k , the partitioning of the ice shelf is updated to include the new crack as

$$L = \{0, x_1, x_2, \dots, x_k, \dots, x_N\}. \quad (13)$$

The eigenvalue problem is re-solved on the new geometry, and the evolution re-initiated from the new initial data

$$W(x, t_k) \mapsto f(x), \quad \partial_t W(x, t_k) \mapsto g(x). \quad (14)$$

This process repeats until the final simulation time is reached.

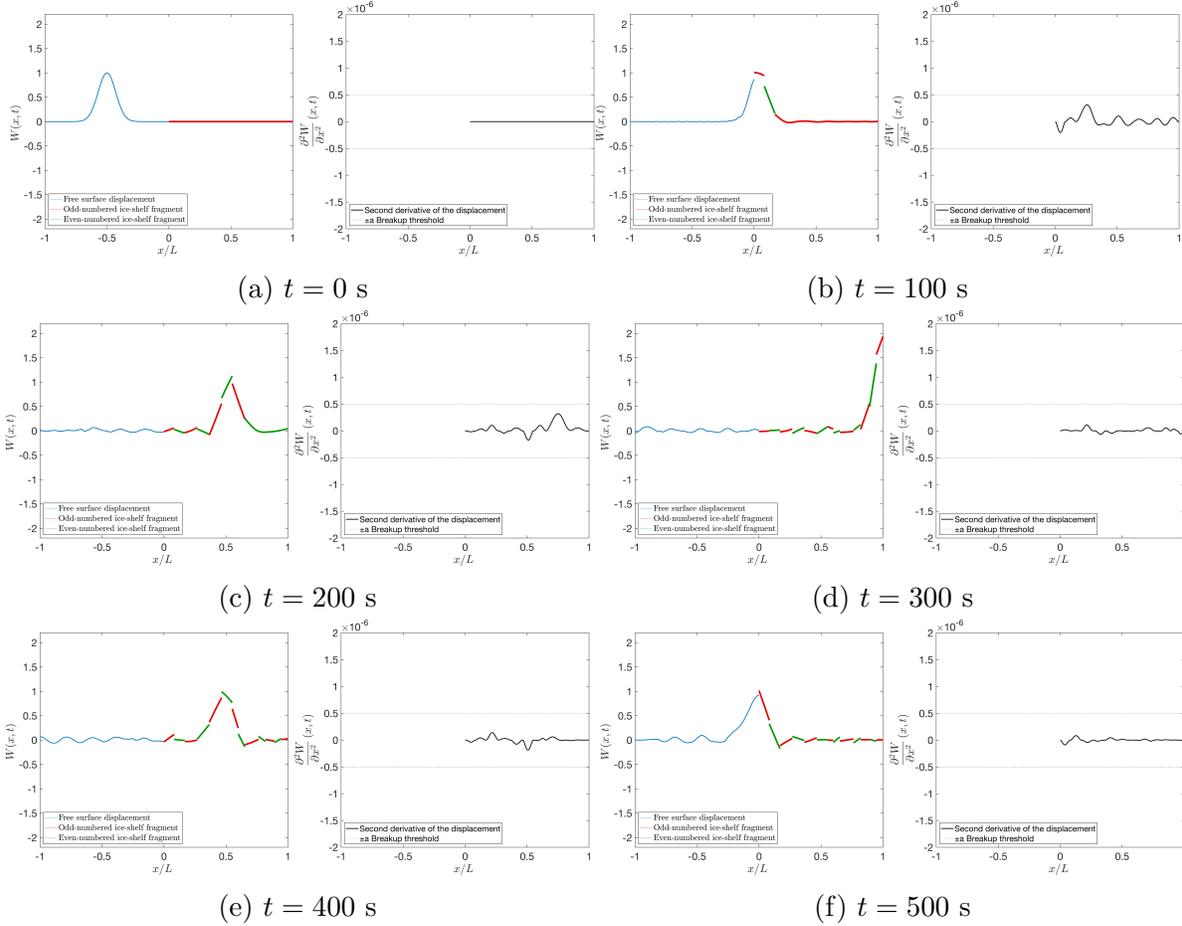


Figure 2: Time-domain simulation of sequential ice-shelf breakup in response to an incident wave pulse. In each subfigure, the left panel shows the systems evolution and the right panel illustrates the local strain criterion.

REFERENCES

- Alshahrani, F., Meylan, M. H., and Wilks, B. (2024). Transient shallow water wave interactions with a partially fragmented ice shelf. *Fluids*, 9(8):192.
- Alshahrani, F., Meylan, M. H., and Wilks, B. (2025). Time-domain analysis of an ice shelf in a bounded domain. *Journal of Fluid Mechanics*. In press.
- Meylan, M. H. (2002). Spectral solution of time-dependent shallow water hydroelasticity. *J. Fluid Mech.*, 454:387–402.
- Sergienko, O. V. (2010). Elastic response of floating glacier ice to impact of long-period ocean waves. *J. Geophys. Res. Earth Surf.*, 115:F04028.
- Vaughan, D. G. (1995). Tidal flexure at ice shelf margins. *J. Geophys. Res. Solid Earth*, 100(B4):6213–6224.