On the interaction of regular waves with linear shear currents by use of the stream function theory

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Highlights

- The model provided by Darlymple (1974) for wave-current interaction shows stability issues. This study provides a more stable numerical algorithm for the interaction of regular waves with a linear shear current.
- Compared with other numerical and experimental results, the accuracy of the proposed solution by the stream function theory for combined regular waves and linear shear currents is relatively high.

1 Introduction

There are various forms of currents in the oceans. In nearshore areas, the interaction between waves and currents poses a threat to marine structures (Wan et al., 2024). Therefore, the study of wave-current coupling is of great significance. This study focuses on the interaction between two-dimensional regular waves and linear shear currents. In terms of physical experiments, Swan (1990) measured the waveform and velocity field of the interaction between regular waves and linear shear currents. Thomas (1990) studied the importance of vorticity in wave-current interaction.

Numerical simulations are typical approaches to study wave-current interaction. Darlymple (1974) proposed the waves and linear shear currents stream function theory (W&LSC stream function theory), and solved it based on the Lagrange multiplier approach and made the computer program openly available (https://www.ce.jhu.edu/dalrymple/). Zhao et al. (2023) proposed a new method for wave-current generation and absorption in numerical tanks using the High Level Green Naghdi (HLGN) model. Fang et al. (2023) proposed fifth-order Stokes wave solution to study fluid particle trajectories under wave-current interaction condition.

The model provided by Darlymple (1974) is unstable and does not converge under strong current conditions (see section 4.2). Therefore, the motivations of this paper are: 1. Propose a stable algorithm applicable to W&LSC stream function theory; 2. Verify the accuracy of modified stream function theory and algorithm. This paper is organized as follows: In section 2, an outline of the W&LSC stream function theory is provided; The new algorithm proposed in this study was introduced in section 3; The numerical results are presented in section 4. Finally, the conclusions were summarized.

2 Wave and linear shear current stream function theory

Darlymple (1974) consider the interaction between regular waves and linear shear currents in two dimensions. Assuming that the fluid is inviscid and incompressible, and that the vorticity is constant. The origin of the Cartesian coordinate system *Oxy* is on the still-water surface and moves forward along the *x*-axis with the wave velocity. The positive direction of the *x*-axis is horizontally to the right, and the positive direction of the *y*-axis is vertically upward, as shown in Figure 1. $y = \eta(x)$ is the free surface, measured from the still-water level. The constant water depth is represented by *h*. The current velocity on the seabed is denoted as U_0.

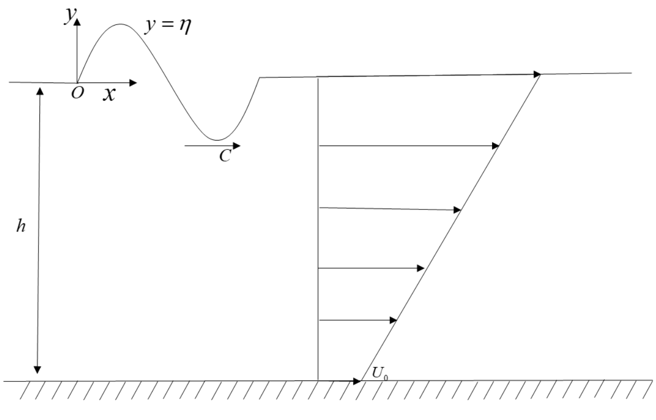


Figure 1 Sketch of the interaction between a regular wave and linear shear current

For an incompressible fluid, the stream function is defined as

$$u(x,y) = -\frac{\partial \psi(x,y)}{\partial y}, \quad v(x,y) = \frac{\partial \psi(x,y)}{\partial x}, \tag{1}$$

where u and v represent the horizontal and vertical velocity components of wave-current interaction in the translational coordinate system, respectively. Dalrymple (1974) proposed the following governing equation and boundary conditions based on the stream function theory:

$$\nabla^2 \psi = -\omega_0, \tag{2a}$$

$$\eta + \frac{1}{2g}[(\frac{\partial \psi}{\partial x})^2 + (\frac{\partial \psi}{\partial y})^2] = Q, \text{ at } y = \eta(x), \tag{2b}$$

$$\psi(x,y) = R, \quad \text{at } y = \eta(x), \tag{2c}$$

where ω_0 is the vorticity, which is constant under linear the shear current condition. Q and R are also constant. Darlymple (1974) assumed the following form for the stream function

$$\psi(x,y) = -(U_0 - \frac{L}{T})y - \frac{\omega_0}{2}(y+h)^2 + \sum_{j=1}^{N} B_j \sinh\frac{2\pi j(h+y)}{L}\cos\frac{2\pi jx}{L}, \tag{3}$$

where L is the wavelength and T is the period. $B_j\,(j = 1,2,\cdots,N)$ is the constant coefficient to be calculated for specific wave-current coupling problem. Substituting equation (3) into equation (2) gives

$$\eta + \frac{1}{2g}\left\{[-\frac{2\pi}{L}\sum_{j=1}^{N} jB_j \sinh\frac{2\pi j(\eta+h)}{L}\sin\frac{2\pi jx}{L}]^2 + [(U_0 - \frac{L}{T}) + \omega_0(\eta+h) - \frac{2\pi}{L}\sum_{j=1}^{N} jB_j \cosh\frac{2\pi j(\eta+h)}{L}\cos\frac{2\pi jx}{L}]^2\right\} = Q, \tag{4a}$$

$$-(U_0 - \frac{L}{T})\eta - \frac{\omega_0}{2}(\eta+h)^2 + \sum_{j=1}^{N} B_j \sinh\frac{2\pi j(\eta+h)}{L}\cos\frac{2\pi jx}{L} = R. \tag{4b}$$

3 Numerical algorithm

Darlymple (1974) solved the wave-current coupling problem by use of the Lagrange multiplier approach and published the developed Fortran program. However, using this approach, the program shows some stability issues. Calculations sometimes fail to converge (see section 4.2). Therefore, in this study, an alternative solution based on the Newton iteration method is used to solve the wave-current interaction equations.

The procedure is as follows: (i) Take half of the wavelength at which the wave current interaction is stable as the calculation length; (ii) Discretize it into N+1 equidistant points to obtain $x_0, x_1, x_2, \cdots, x_N$, where $x_i = i\Delta x = iL/2N$ ($i = 0,1,\cdots,N$); (iii) Define the wave profile as $\eta_0, \eta_1, \eta_2, \cdots, \eta_N$, where ($x_0$, η_0) is the point at the wave crest and (x_N , η_N) is the point at the wave trough. Then equation (4) is written as

$$\eta_i + \frac{1}{2g}\left\{[-\frac{2\pi}{L}\sum_{j=1}^{N} jB_j \sinh\frac{2\pi j(\eta_i+h)}{L}\sin\frac{ji\pi}{N}]^2 + [(U_0 - \frac{L}{T}) + \omega_0(\eta_i+h) - \frac{2\pi}{L}\sum_{j=1}^{N} jB_j \cosh\frac{2\pi j(\eta_i+h)}{L}\cos\frac{ji\pi}{N}]^2\right\} = Q, \quad i = 0,1,2\cdots,N, \tag{5a}$$

$$-(U_0 - \frac{L}{T})\eta_i - \frac{\omega_0}{2}(\eta_i+h)^2 + \sum_{j=1}^{N} B_j \sinh\frac{2\pi j(\eta_i+h)}{L}\cos\frac{ji\pi}{N} = R, \quad i = 0,1,2\cdots,N. \tag{5b}$$

Following this procedure, there are 2N+2 equations with 2N+4 unknowns, denoted as $\eta_0, \eta_1, \eta_2, \cdots, \eta_N$, $B_1, B_2, \cdots, B_N$, L, R, Q. Therefore, two additional equations are required to close the system of equations.

There is no change in the mean water level (Darlymple, 1974). That is, $\frac{2}{L}\int_0^{\frac{L}{2}} \eta(x)\mathrm{d}x = 0$, expressed in discrete form as

$$\frac{1}{2N}[\eta_0 + \eta_N + 2\sum_{j=1}^{N-1}\eta_j] = 0. \tag{6}$$

Furthermore, the relationship between the wave crest and wave trough leads to

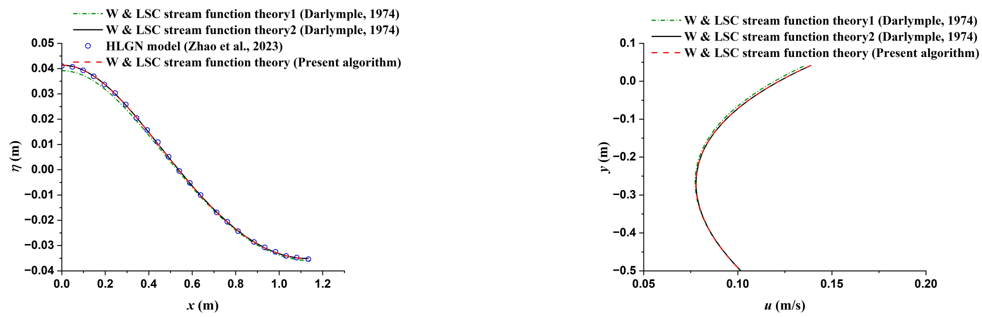
$$\eta_0 - \eta_N = H, \tag{7}$$

where H is the wave height. The system of equations (5)-(7) consists of 2N+4 equations and 2N+4 unknowns. The closed system of equations can be solved using Newton's iterative method.

4 Numerical results

4.1 Case1: Zhao et al. (2023)

The interaction between regular waves and opposing linear shear current is studied using the model provided by Darlymple (1974) and the new algorithm proposed in this paper. Zhao et al. (2023) analyzed the coupling between regular waves and opposing current with u_c = –0.2y–0.1m/s by use of the HLGN model. The water depth is set to h=0.5m. The wave height is H=0.07655m, and the period is T=1.325s. The wave profile and velocity field of the stable solution can be obtained using the Newton iteration method in this study, as shown in Figure 2. We use N=50 in the calculations, and the same applies to the subsequent calculations.



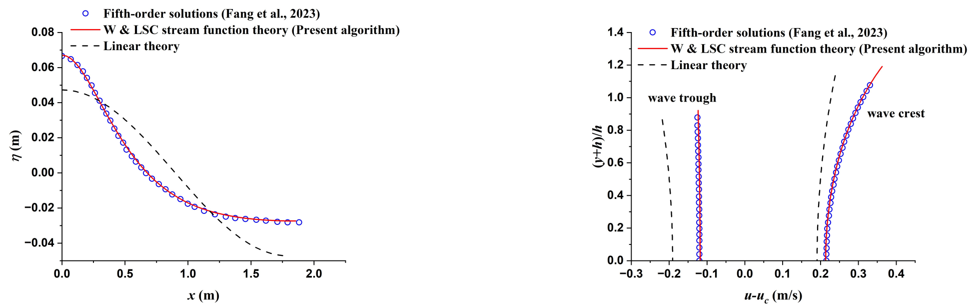
(a) Wave profile (b) Horizontal velocity under the wave crest

Figure 2 Regular wave interacting with a linear shear current $u_c = -0.2y-0.1$m/s (H=0.07655m, h=0.5m, T=1.325s)

The W&LSC stream function theory1 in Figure 2 represents the results calculated using the model provided by Darlymple (1974). The tolerance setting in the model is 0.01. This study found that the calculation accuracy is low for this case. Therefore, we reduce the tolerance to 0.00001 to modify the model. The result obtained is presented as W&LSC stream function theory 2. There is a significant difference between the results before and after modifying the model. The W&LSC stream function theory 2 is an accurate result of the model provided by Darlymple (1974). The calculation results of W&LSC stream function theory 2 are completely consistent with the algorithm presented in this paper. This indicates that the new algorithm proposed in this study can accurately simulate the waveform and velocity field of wave-current coupling problems. In addition, Figure 2 (a) shows that the waveform obtained by our algorithm is consistent with the waveform calculated by Zhao et al. (2023), which further demonstrates the accuracy of our algorithm.

4.2 Case2: Fang et al. (2023)

We have found that the model developed by Darlymple (1974) is unable to obtain convergent solutions for some wave-current coupling problems, while the algorithm proposed in this paper overcomes the issue and can obtain converged results. For example, Fang et al. (2023) explored the interaction between regular waves and strong following linearly sheared current $u_c = 1.7y+1.0325$ with water depth h=0.35m, wave height H=0.0945m, and period T=1.418s. In this study, the algorithm proposed in this paper was used for the calculations. The wave profile obtained by this algorithm and Fang et al. (2023) is shown in Figure 3 (a). The vertical distribution of $u-u_c$ under the wave crest and wave trough are shown in Figure 3 (b). The linear theory results are also presented in the figures to see the differences between nonlinearity and linearity.



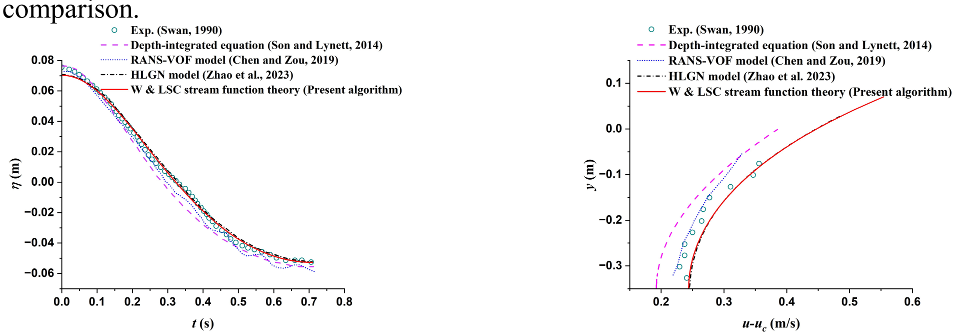
(a) Wave profile (b) $u-u_c$ under the wave crest and wave trough

Figure 3 Regular wave interacting with a linear shear current $u_c = 1.7y+1.0325$m/s (H=0.0945m, h=0.35m, T=1.418s)

Figure 3 shows that the wave profile and $u-u_c$ under the wave crest and wave trough obtained by the present algorithm and Fang et al.'s (2023) fifth order solution are in complete agreement. The phenomenon of sharp wave crest and flat trough is obvious, which is different with the results of linear theory, shown in Fig. 3. There is also a significant difference between $u-u_c$ under the wave crest calculated by present algorithm and the result by Fang et al. (2023). According to Figure 3 (a), the wave crest position is y=0.067m. The water depth is h=0.35m in this case, then $(y+h)/h$ =1.19.

4.3 Case3: Swan (1990)

Next we consider the laboratory measurements by Swan (1990), who studied the interaction between progressive wave train and shear current $u_c = -1.67y-0.5$. In the experiments, the water depth was 0.35m, the wave height was 0.123m, and the period was 1.42s. The average value of the experimental wave profile is $\int_0^{T/2} \eta(t)\mathrm{d}t = -0.0072$. However, since the model assumes that the average surface elevation is zero, the laboratory measurements from Swan (1990) are adjusted by shifting the data up by 0.011m to meet this condition. We numerically reproduce the experimental results of Swan (1990) using the algorithm proposed in this paper. The wave profile result is shown in Figure 4 (a). The experimental data in the figure is shifted up by 0.011m, but the numerical results have not shifted up. The $u-u_c$ under the wave crest measured in the experiment and the calculated results of our algorithm are shown in Figure 4 (b). We also include results from Son and Lynett (2014), Chen and Zou (2019) and Zhao et al. (2023) in this comparison.



(a) Wave profile (b) $u-u_c$ under the wave crest

Figure 4 Regular wave interacting with a linear shear current $u_c = -1.67y-0.5$m/s (H=0.123m, h=0.35m, T=1.42s)

On the wave profile, Figure 4 (a) shows that the calculation results of our algorithm are in good agreement with the experimental data. The wave profile calculated by Chen and Zou (2019), obtained by the RANS approach, is not smooth at the wave trough. The results of Son and Lynett (2014), using depth-integrated equations, show a significant deviation from the experimental data. On the $u-u_c$ under the wave crest, the calculation results of our algorithm are in good agreement with the experimental data, as shown in Figure 4 (b). The results of this paper are in very good agreement with those of Zhao et al. (2023), with only slight differences observed near the seafloor. The calculation results of Chen and Zou (2019) differ significantly from experimental data near the seafloor and the free surface. The $u-u_c$ calculated by Son and Lynett (2014) is significantly smaller than the experimental data.

5 Conclusions

The approach and code provided by Darlymple (1974) to study the interaction between waves and linear shear currents shows instability under extreme conditions, with the calculations not converging. This paper proposes a new algorithm using Newton's iterative method based on the stream function theory. The model converges at conditions where original model has failed. Results of the model are compared with the numerical results of Zhao et al. (2023), the fifth order analytical solution of Fang et al. (2023), and the experimental data of Swan (1990), and the original W&LSC stream function theory, demonstrating the convergence and accuracy of the results.

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