Wave interaction with a large number of floating ice sheets of arbitrary shapes: exact and wide space-approximation approaches

Yifeng Yang[1], **Guoxiong Wu**[1]
[1]Department of Mechanical Engineering, University College London, London, UK.
E-mail Address of the presenting author: yifeng.yang.19@ucl.ac.uk

1 INTRODUCTION

The marginal ice zone (MIZ) is the transitional zone between open sea and expansive floating ice plane, and one of such cases is characterised by a large number of discrete floating ice sheets [1]. These ice sheets engage in complex interaction with waves, which can significantly influence the dynamic evolution of the icy water environment. To understand such interactions, a series of studies have been conducted in the past decades. For example, a semi-analytical procedure is proposed for wave scattering by multiple circular ice floes [2], and an approximation method is used for circular ice floes in random distribution [3]. Here, we shall consider the problem of wave interaction with multiple floating ice sheets of arbitrary shapes, based on a hybrid approach that combines the eigenfunction expansion with the boundary element method (BEM). A key challenge arises when dealing with a large number of ice sheets, as the direct computation becomes highly time-consuming. To overcome this difficulty, a wide-space approximation is adopted, where each ice sheet is treated individually, and only the effect of its traveling wave on other ice sheets is considered, which improves the computational efficiency drastically.

2 MATHEMATICAL MODELLING AND SOLUTION PROCEDURE

The problem of wave interaction with N_I floating ice sheets of arbitrary shapes is sketched in Figure 1. The i-th ice sheet, $i = 1 \sim N_I$, is assumed to have homogeneous physical properties, with density ρ_i and thickness h_i. To describe the problem, a Cartesian coordinate system O-xyz is established with the origin located at the mean water surface, the x-axis and y-axis are along the transverse plane, and the z-axis points upward.

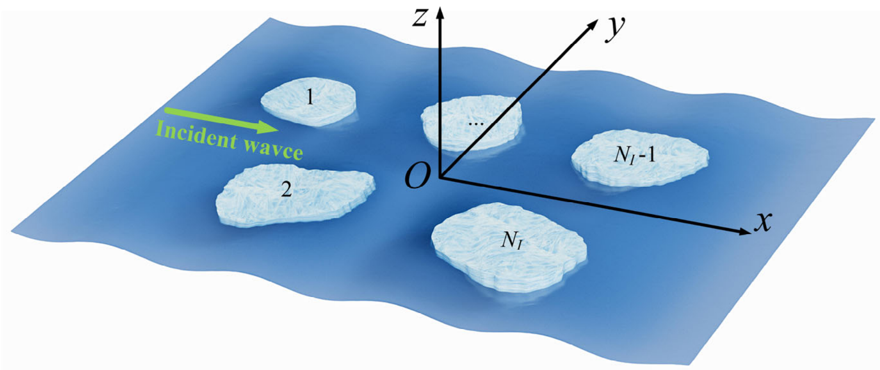

Figure 1: Sketch of wave interaction with a large number of floating ice sheets

The fluid of mean water depth H and density ρ is assumed to be homogeneous, inviscid and incompressible, and its motion is irrotational. Hence, the fluid flow can be described by the velocity potential Φ. For small amplitude waves, linearization is further introduced. For a sinusoidal wave in time with frequency ω, Φ can be written as

$$\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, y, z) \times e^{-\mathrm{i}\omega t}\}, \tag{1}$$

where $\phi(x, y, z)$ is due to the incident and diffracted waves. ϕ satisfies the Laplace equation throughout the fluid domain

$$\nabla^2\phi + \frac{\partial^2\phi}{\partial z^2} = 0, \qquad -\infty < x, y < +\infty, -H \leq z \leq 0, \tag{2}$$

where ∇^2 denotes the two-dimensional Laplacian in the O-xy plane. All the ice sheets are assumed to float on the water surface with the draught effect neglected. The boundary conditions on the lower surface of the ice sheet S_i $(i = 1\sim N_I)$ are imposed at $z = 0$

$$(L_i\nabla^4 - m_i\omega^2 + \rho g)\frac{\partial\phi}{\partial z} - \rho\omega^2\phi = 0, \qquad (x, y) \in S_i\ (i = 1\sim N_I), \tag{3}$$

where $L_i = E_i h_i^3/[12(1 - \nu_i^2)]$, E_i and ν_i represent the flexural rigidity, Young's modulus and Poisson's ratio of the i-th ice sheet respectively, $m_i = \rho_i h_i$ denotes the mass per unit area of the ice sheet, and g represents the acceleration due to gravity. In the free surface, the combined dynamic and kinematic boundary conditions provide

$$g\frac{\partial\phi}{\partial z} - \omega^2\phi = 0, \qquad z = 0, (x, y) \notin S_i\ (i = 1\sim N_I). \tag{4}$$

The impermeable boundary condition on the flat seabed can be expressed as

$$\partial\phi/\partial z = 0, \qquad z = -H. \tag{5}$$

At the far-field, the radiation condition must be imposed to ensure waves propagating outward. Besides, the edge of each ice sheet is assumed to be free to move, which requires the zero-bending moment and shear force conditions to be imposed, or [4]

$$\mathcal{B}\left(\frac{\partial\phi}{\partial z}\right) = 0 \quad \text{and} \quad \mathcal{S}\left(\frac{\partial\phi}{\partial z}\right) = 0, (x, y) \in \Gamma_i\ (i = 1\sim N_I), \tag{6}$$

where $\Gamma_i = \big(x(s), y(s)\big)$ represents the edge of S_i, and s is the curvilinear coordinate along the edge. The operators $\mathcal{B}$ and $\mathcal{S}$ are defined as

$$\mathcal{B} = \nabla^2 - \nu_i'\left(\frac{\partial^2}{\partial s^2} + \frac{\partial\Theta}{\partial s}\frac{\partial}{\partial n}\right) \quad \text{and} \quad \mathcal{S} = \frac{\partial}{\partial n}\nabla^2 + \nu_i'\frac{\partial}{\partial s}\left(\frac{\partial^2}{\partial s\partial n} - \frac{\partial\Theta}{\partial s}\frac{\partial}{\partial s}\right). \tag{7a, b}$$

with $\nu_i' = 1 - \nu_i$, $\boldsymbol{n} = (\cos\Theta, \sin\Theta)$ and $\boldsymbol{s} = (-\sin\Theta, \cos\Theta)$ denote the unit normal and tangential vectors of Γ_i respectively.

To solve the boundary value problem above, we may define ϕ_i as the velocity potential of the fluid motion beneath the i-th ice sheet. ϕ_i can be constructed from the boundary integral equation [5]

$$\Lambda_1(\boldsymbol{P})\phi_i(\boldsymbol{P}) = \int_{\Gamma_i}\left[\langle\phi_i(\boldsymbol{Q}), \frac{\partial G_i(\boldsymbol{P},\boldsymbol{Q})}{\partial n}\rangle - \langle G_i(\boldsymbol{P},\boldsymbol{Q}), \frac{\partial\phi_i(\boldsymbol{Q})}{\partial n}\rangle\right]\mathrm{d}s, \qquad i = 1\sim N_I \tag{8}$$

where $\boldsymbol{P}(x, y, z)$ and $\boldsymbol{Q}(\xi, \eta, \zeta)$ represent the field and source points respectively, Λ_1 is the solid angle at $\boldsymbol{P}$. G_i is the Green function for the fluid fully covered by the i-th ice sheet. Following the procedure in Li et al. [6], in the interface $S_c^{(i)}$, ϕ_i and $\partial\phi_i/\partial n$ on $S_c^{(i)}$ can be expanded into eigenfunction series as,

$$\phi_i(\boldsymbol{P}) = \sum_{m=-2}^{+\infty}\varphi_m^{(i)}(x, y)\psi_m^{(i)}(z) \quad \text{and} \quad \frac{\partial\phi_i(\boldsymbol{P})}{\partial n} = \sum_{m=-2}^{+\infty}\frac{\partial\varphi_m^{(i)}(x,y)}{\partial n}\psi_m^{(i)}(z), \tag{9a, b}$$

where $\psi_m^{(i)}(z) = \cosh\left[\kappa_m^{(i)}(z + H)\right]/\cosh\left(\kappa_m^{(i)}H\right)$, $\kappa_m^{(i)}$ are the roots of the dispersion equation $K_i(\alpha, \omega) = (L_i\alpha^4 - m_i\omega^2 + \rho g)\alpha\tanh(\alpha H) - \rho\omega^2 = 0$. Particularly, $\kappa_0^{(i)}$ is a purely positive real root, $\kappa_{-1}^{(i)}$ and $\kappa_{-2}^{(i)}$ are two complex roots with negative imaginary parts, with $\kappa_{-1}^{(i)} = -\bar{\kappa}_{-2}^{(i)}$. $\kappa_m^{(i)}$ $(m = 1, 2\ldots)$ are an infinite number of purely negative imaginary roots. Substituting Eqs. (9a, b) into Eq. (8), we obtain

$$\Lambda_1(\boldsymbol{P})\phi_i(\boldsymbol{P}) = \pi\mathrm{i}\sum_{m=-2}^{+\infty}\psi_m^{(i)}(z)\int_{\Gamma_i}\left[\varphi_m^{(i)}(\xi, \eta)\frac{\partial\mathcal{H}_0^{(2)}\left(\kappa_m^{(i)}R\right)}{\partial n} - \frac{\partial\varphi_m^{(i)}(\xi,\eta)}{\partial N}\mathcal{H}_0^{(2)}\left(\kappa_m^{(i)}R\right)\right]\mathrm{d}s, \tag{10}$$

where $R = \sqrt{(x - \xi)^2 + (y - \eta)^2}$, $\mathcal{H}_0^{(2)}(\alpha)$ denotes the 0-th order Hankel function of the second kind.

We may write the velocity potential in the free surface region as $\phi = \phi_I + \phi_D$, where the subscripts I and D denote the incident and diffracted components respectively. ϕ_D is also constructed using the boundary integral equation method, which provides

$$\Lambda_2(\boldsymbol{P})\phi_D(\boldsymbol{P}) = \pi \mathrm{i} \sum_{i=1}^{N_I} \sum_{m=0}^{+\infty} Z_m(z) \int_{\Gamma_i} \left[\phi_m^{(i)}(\xi,\eta) \frac{\partial \mathcal{H}_0^{(2)}(k_m R)}{\partial n} - \frac{\partial \phi_m^{(i)}(\xi,\eta)}{\partial N} \mathcal{H}_0^{(2)}(k_m R)\right] \mathrm{d}s, \quad (11)$$

where $Z_m(z) = \cosh[k_m(z+H)] / \cosh(k_m H)$, $\phi_m^{(i)}$ and $\partial\phi_m^{(i)}/\partial n$ are the functions in Eq.(9), k_m are the roots of the dispersion equation $K_w(\alpha,\omega) = g\alpha \tanh(\alpha H) - \omega^2 = 0$. k_0 is a purely positive real root, k_m $(m = 1, 2 \ldots)$ are an infinite number of purely negative imaginary roots. For incoming wave along the x direction, ϕ_I can be written as

$$\phi_I(\boldsymbol{P}) = I Z_0(z) e^{\mathrm{i} k_0 x}. \quad (12)$$

where $I = Ag/\mathrm{i}\omega$, A is the amplitude of the wave. To solve the problem, we may let the field point $\boldsymbol{P}$ in Eqs. (10) and (11) be located on $S_c^{(i)}$. Multiplying both sides with $\psi_m^{(i)}(z)$ and $Z_m(z)$ separately, and using the orthogonal properties, we have

$$\frac{\Lambda_1(\boldsymbol{P})}{\pi \mathrm{i}} \varphi_m^{(i)}(x,y) - \int_{\Gamma_i} \left[\varphi_m^{(i)}(\xi,\eta) \frac{\partial \mathcal{H}_0^{(2)}\left(\kappa_m^{(i)} R\right)}{\partial n} - \frac{\partial \varphi_m^{(i)}(\xi,\eta)}{\partial N} \mathcal{H}_0^{(2)}\left(\kappa_m^{(i)} R\right)\right] \mathrm{d}s = 0, \quad (13)$$

$$\frac{\Lambda_2(\boldsymbol{P})}{\pi \mathrm{i}} \phi_m^{(i)}(x,y) - \sum_{j=1}^{N_I} \int_{\Gamma_i} \left[\phi_m^{(j)}(\xi,\eta) \frac{\partial \mathcal{H}_0^{(2)}(k_m R)}{\partial n} - \frac{\partial \phi_m^{(j)}(\xi,\eta)}{\partial N} \mathcal{H}_0^{(2)}(k_m R)\right] \mathrm{d}s = 0. \quad (14)$$

The remaining equations can be obtained by matching the velocity potential and normal velocity at each vertical control surface $S_c^{(i)}$, and the edge conditions at the ice sheet. This can be imposed by using the orthogonal product of $\psi_m^{(i)}(z)$, which provides

$$Q_m^{(i)} \varphi_m^{(i)}(x,y) - \sum_{m'=0}^{+\infty} E_{m,m'} \phi_{m'}^{(i)}(x,y) - \sum_{m'=-2}^{+\infty} f_{m,m'}^P \varphi_{m'}^{(i)}(x,y) = I E_{m,0} e^{\mathrm{i} k_0 x}, \quad (15)$$

$$Q_m^{(i)} \frac{\partial \varphi_m^{(i)}(x,y)}{\partial n} - \sum_{m'=0}^{+\infty} E_{m,m'} \frac{\partial \phi_{m'}^{(i)}(x,y)}{\partial n} - \sum_{m'=-2}^{+\infty} f_{m,m'}^V \varphi_m^{(i)}(x,y) = I E_{m,0} \frac{\partial e^{\mathrm{i} k_0 x}}{\partial n}. \quad (16)$$

where $Q_m^{(i)}$, $E_{m,m'}$ are known coefficients and $f_{m,m'}^P$ and $f_{m,m'}^V$ are known operators [6].

We may divide the ice edge Γ_i $(i = 1 \sim N_I)$ into a sufficient number of straight-line segments. For each segment, $\varphi_m^{(i)}$, $\partial\varphi_m^{(i)}/\partial n$, $\phi_m^{(i)}$ and $\partial\phi_m^{(i)}/\partial n$ are assumed to be constant and are taken as the values at the centre of the segment. In such a case, Eqs. (13) ~ (16) can be solved numerically using the panel method.

For cases with a large number of floating ice sheets, the direct solution from Eqs. (13) ~ (16) becomes very inefficient or impractical. To overcome this, a wide-space approximation is proposed here. We note that the mode $m = 0$ in the eigenfunction expansion corresponds to the traveling wave, while terms of $m \neq 0$ represent evanescent waves, which decay exponentially. Therefore, in Eq. (14), when the source point $\boldsymbol{Q}$ is located on the other ice sheets, or $j \neq i$, only the $m = 0$ mode, or the first few terms, are retained. These terms are then moved to the right-hand side of the equations. The solution for each ice sheet is first obtained on its own. Then the travelling wave term or the first few terms is put on the right-hand side and the equation for each ice sheet is resolved and the solution is corrected. The computational time is drastically reduced.

3 RESULTS AND ANALYSIS

The following physical parameters are used in the computation: $\rho = 1025\ \mathrm{kg/m^3}$, $\rho_i = 922.5\ \mathrm{kg/m^3}$, $E_i = 5$ GPa, $\nu_i = 0.3$ and $H = 50$ m. A case study for 4 identical floating circular ice floes with radius $a = 10$ m is conducted. The centres of the floes are located at $x_i = 16(i-1)a$ and $y_i = 0$ $(i = 1 \sim 4)$. The force and moment on the first and fourth ice floes are given in Figures 2 and 3. A good agreement is found between the results from the exact numerical solution and the wide-space approximation.

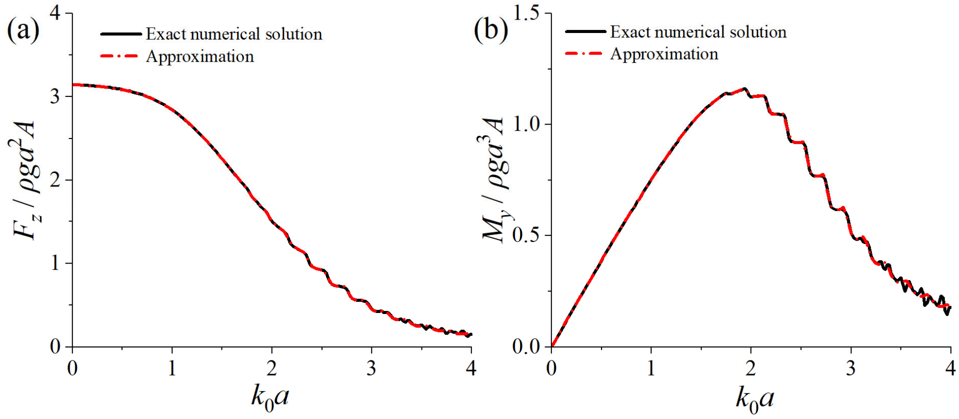

Figure 2: Force (a) and moment (b) on the 1st circular ice sheet ($H/a = 5$, $h_i/a = 0.05$).

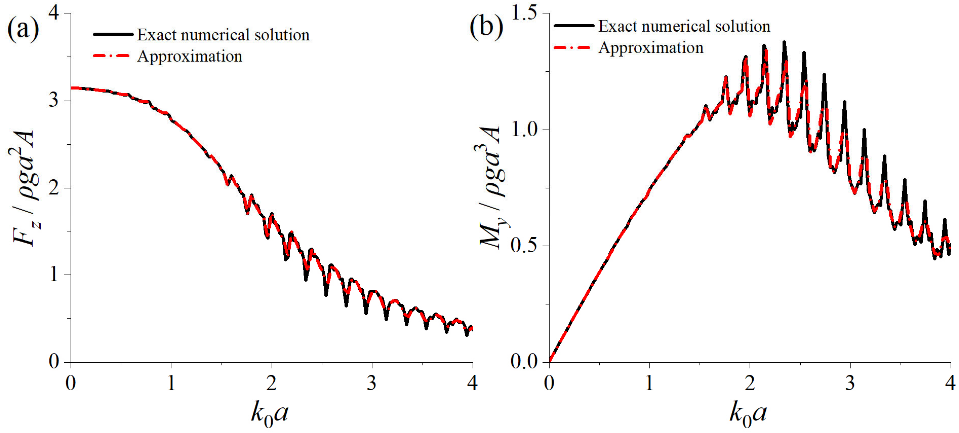

Figure 3: Force (a) and moment (b) on the 4th circular ice sheet ($H/a = 5$, $h_i/a = 0.05$).

4 CONCLUSIONS

The interaction of waves and multiple floating ice sheets of arbitrary shapes are studied based on a hybrid method, combining vertical mode expansion and BEM. A wide-space approximation procedure is also proposed to improve the computational efficiency. Results for arbitrary shapes and a very large number of ice sheets will be presented in the workshop.

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