Chaotic Dynamic Characteristics during the Long Time Evolution of Wave Trains

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1 INTRODUCTION

The long time evolution process of modulated wave trains contains rich nonlinear dynamic characteristics. Previous studies have shown that there are two important nonlinear mechanisms in the long time evolution of wave trains, namely modulation instability and nonlinear wave group interactions [1,2]. Investigating its chaotic dynamic characteristics not only helps to reveal the intrinsic mechanisms of state transitions in wave systems, but also has significant implications for predicting the evolution trends of wave fields. By quantitatively analyzing the nonlinear features in the time series of wave surface, it is possible to identify the stability changes of the wave field, evaluate the time scale of predictions, and provide essential support for understanding the fundamental characteristics of nonlinear wave systems and predicting the evolution of wave fields. Therefore, the main purpose of the study is to qualitatively and quantitatively analyze and evaluate the chaotic dynamic characteristics during the long time evolution of wave trains based on the relevant knowledge of chaos theory, through Phase space reconstruction, Largest Lyapunov Exponent, Correlation Dimension, and Kolmogorov Entropy.

2 METHODOLOGY

The study employs the High-Order Spectral (HOS) method, which is a phase-resolved numerical model used for simulating water wave evolution processes. The model was first proposed by Dommermuth and Yue [3], and West [4] in 1987, respectively. Based on the Zakharov equation and the concepts of harmonic coupling method, and incorporating the advantages of the Fast Fourier Transform (FFT) algorithm.

The initial condition of the study is the modulated wave train, which consists of a carrier wave and a pair of sidebands equidistant from the carrier wave. The initial conditions are as follows:

$$\left.\begin{aligned} \eta(x,0) &= \eta_0\left[\varepsilon_0,k_0\right] + r_{21}a_0\cos(k_+x-\theta_+) + r_{22}a_0\cos(k_-x-\theta_-) \\ \varphi^s(x,0) &= -\varphi_0^s\left[\varepsilon_0,k_0\right] + \frac{r_{21}a_0}{\sqrt{k_+}}e^{k_+\eta}\sin(k_+x-\theta_+) + \frac{r_{22}a_0}{\sqrt{k_-}}e^{k_-\eta}\sin(k_-x-\theta_-) \end{aligned}\right\} \tag{1}$$

where $\eta(x, 0)$ and $\phi^s(x, 0)$ represent the wave surface elevation and the velocity potential at the initial time, respectively. These correspond to the free surface elevation and surface velocity potential of the carrier wave and can be directly calculated using the analytical solutions of high-order Stokes waves, as proposed by Schwartz (1974) [5]. Here, ε_0 and k_0 denote the steepness and wavenumber of the carrier wave, respectively, where $\varepsilon_0=k_0a_0$, and a_0 is the amplitude of the carrier wave. The sideband wavenumbers and phases are given by $k_\pm=k_0\pm\Delta k$ and $\theta_\pm=\theta_0\pm\Delta\theta$, where Δk is the difference between the sideband wavenumber and the carrier wave wavenumber, θ_0 is the phase of the carrier wave, and $\Delta\theta$ is the phase difference between the sideband and the carrier wave. The amplitudes of the upper and lower sidebands are given by a_+ and a_-, respectively, and their ratios to the carrier wave amplitude are defined as $r_{21}=a_+/a_0$ and $r_{22}=a_-/a_0$. The parameters for the HOS method are set as M=7, N=4096, T_0/dt=64, r_{21}=r_{22}=0.1, where T_0 is the period of the carrier wave and dt is the time step. The simulation time duration is approximately $t/T_0 \approx 2O(\varepsilon_0^{-3})$. And the initial wavenumbers–amplitude spectral as well as the corresponding wave surface of initial modulated wave train are presented in Fig.1 under the condition of ε_0=0.06.

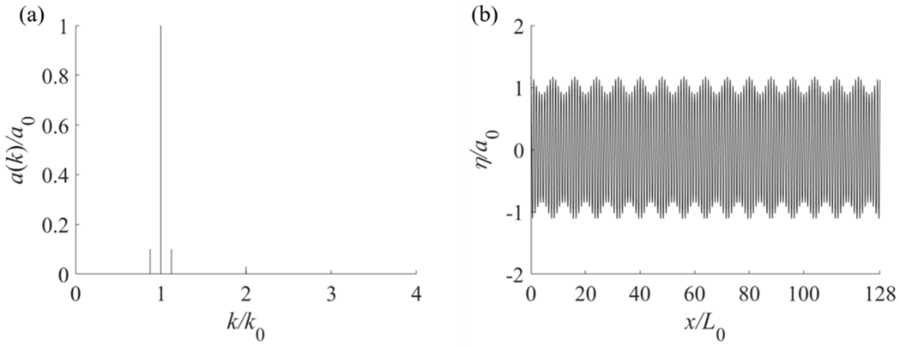

Fig.1 (a) Initial wavenumber–amplitude spectral and (b) Initial wave surface under the condition of ε_0=0.06

3 EVOLUTION OF WAVE SURFACE

The evolution of wave surface is the basis for analyzing the chaotic dynamic characteristics of wave trains during long time evolution. Fig.3 illustrates the evolution process of the wave surface maximum under the condition of ε_0=0.06. In the figure, the horizontal axis, t/T_0, represents the time scale of the evolution, while the vertical axis denotes the maximum spatial wave surface at any given moment during the evolution, expressed as the dimensionless parameter η_{max}/a_0.

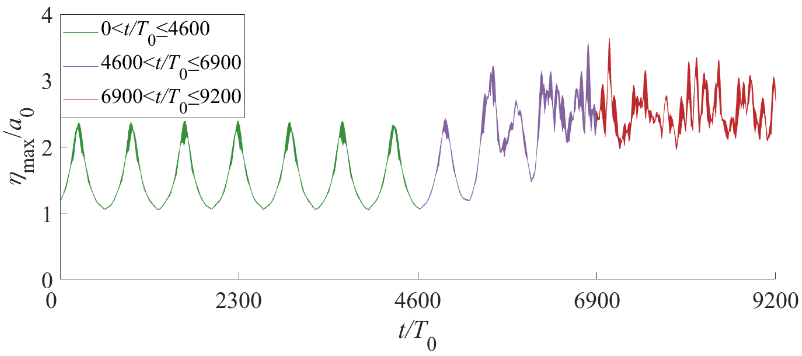

Fig. 3 Evolution of wave surface maximum for initial wave steepness ε_0=0.06

Taking ε_0=0.06 as an example, according to the evolution of wave surface maximum, the long time evolution process ($2O(\varepsilon_0^{-3})T_0$) can be divided into three distinct stages, that is, the stage of modulation instability (green line), the transition stage (purple line), and the stage of nonlinear wave group interaction (red line) [2]. During the stage of modulation instability, the evolution of wave surface maximum exhibits periodic modulation-demodulation phenomena due to the instability of the modulated wave train. As the evolution progresses to the transition stage, the periodic variation of the wave surface maximum gradually weakens, while the wave surface shows an increasing trend. This is caused by the enhancement of nonlinearity and the gradual weakening of modulation instability as the dominant mechanism. In the stage of wave group interaction, the fluctuation amplitude of wave surface maximum increases further, and the fluctuation frequency becomes higher.

4 ANALYSIS ON CHAOTIC DYNAMIC CHARACTERISTICS OF WAVE TRAINS

The phase space is a conceptual tool used to describe dynamical systems, and it contains all possible states of the system [6]. As shown in Fig.4, the reconstruction trajectories exhibit distinct stage characteristics, and the entire process can be divided into three parts: the closed triangle trajectory in the lower left corner (green line), the irregular finite region in the upper right corner (red line), and the transition trajectory connecting the two regions (purple line). These three parts correspond to the stage of modulation instability, nonlinear wave group interaction, and the transition stage in the long time evolution of wave trains, respectively.

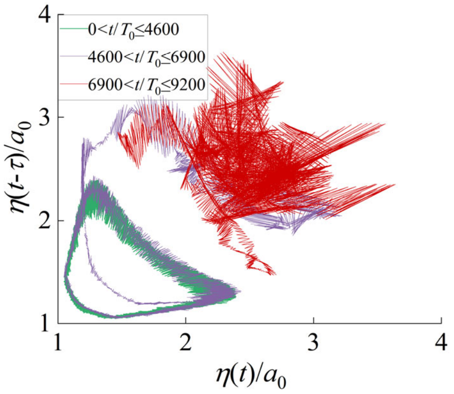

Fig.4 Two-Dimensional phase space reconstruction trajectories for initial wave steepness ε_0=0.06

The triangular region (green line) corresponds to the stage of modulation instability. After a finite time of evolution, the reconstruction trajectories form a closed region. Such closed trajectories typically indicate repetitive behavior of the system, suggestingt that the system state returns to its initial condition over time. This repetitive phenomenon of the reconstruction trajectories corresponds precisely to the periodic recurrence of modulation and demodulation of the wave surface during the stage of modulation instability. At this stage, the wave field remains in a stable state. In the transition stage (purple line), the spatial reconstruction trajectory shows a phenomenon of moving away from the triangular enclosed area, indicating that the attractor state of the wave field gradually changes, transitioning from a triangular attractor to a strange attractor. And the wavefield transitions from stable and predictable at the beginning to unstable. In the stage of nonlinear wave group interaction (red line), unlike the stage of modulation instability, the reconstruction trajectories at this stage are no longer regular and orderly, nor are they concentrated in a specific region. Instead, they appear disordered and intertwined, primarily concentrated within an irregular finite region, forming an abnormal and complex attractor known as a strange attractor, this is an important characteristic of chaotic systems, indicating that chaotic behavior is highly likely to exist during this stage.

Based on the qualitative analysis, we have preliminarily identified that the wave field transitions from a stable state to a chaotic state during the long time evolution of wave trains. Furthermore, the results of the Correlation Dimension (CD), Largest Lyapunov Exponent (LLE), and the Kolmogorov Entropy (KE) at different time scales under distinct wave steepness ε_0 conditions are shown in Table 1. It is evident that under different wave steepness ε_0 conditions, when the wave train evolution time is relatively short ($O(\varepsilon_0^{-2})T_0$), all LLE values are negative, indicating that the wave field remains a stable state within this time scale. During this stage, there are no significant differences in CD values, and KE show a positive correlation with ε_0, suggesting that stronger nonlinearity weakens the stability of wave field. As wave trains further evolves to

$O(\varepsilon_0^{-3})T_0$, we found that at lower initial wave steepness (less than 0.08), LLE remains negative, indicating a stable wave field state. When ε_0 increases, LLE>0. Although still in the stage of modulation instability, due to stronger nonlinearity, the wave field under larger wave steepness conditions transitions from an initially stable state to a chaotic state. During this stage, there are also no significant differences in CD values, and overall, KE values show an increasing trend with the growth of ε_0. These results demonstrate that both initial wave steepness and evolution time length significantly influence the wave field state, and the greater the wave steepness, the less stable the wave field becomes.

Table.1 Chaos parameters at different time scales under distinct wave steepness ε_0 conditions

ε_0	$O(\varepsilon_0^{-2})T_0$				$O(\varepsilon_0^{-3})T_0$			
	LLE	CD	KE	State	LLE	CD	KE	State
0.05	-0.024	1.33	0.019	Stable	-0.00081	1.21	0.0015	Stable
0.06	-0.039	1.13	0.021	Stable	-0.0034	1.40	0.0031	Stable
0.07	-0.069	1.07	0.034	Stable	-0.0050	1.14	0.0029	Stable
0.08	-0.019	1.57	0.038	Stable	0.015	1.12	0.0061	Chaotic
0.09	-0.10	1.11	0.10	Stable	0.011	1.11	0.0046	Chaotic

Additionally, we adopt Largest Lyapunov Exponent (LLE) to investigate the influence of the evolution time length on the state of the wave field, as shown in Fig.5. It can be clearly observed that before $t/T_0 \approx O(\varepsilon_0^{-3})$, that is, in the stage of modulation instability, LLE<0, indicating a stable wave field state. As the evolution time increases, the value of LLE gradually increases. When it evolves to $t/T_0 \approx 3O(\varepsilon_0^{-3})$, LLE>0, indicating that the wave field is completely in a chaotic state. This result further confirms that the long time evolution of wave trains undergoes a transition process from a stable state to a chaotic state.

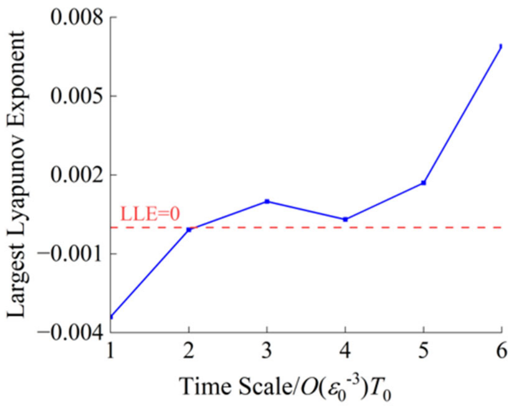

Fig.5 Largest Lyapunov Exponents at different evolution time scales for ε_0=0.06

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