Oscillatory line dipole for water waves in the presence of a shear layer

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1 INTRODUCTION

The linear problem of oscillation of a body submerged in a free-surface fluid has been studied well enough [1,2]. It was assumed that in an undisturbed state the fluid is either at rest or flows at a constant velocity over depth. However, under real conditions there is often a variation in the velocity and the direction of the fluid current with depth. The review of studies on the interaction of surface waves and shear flows was given in [3,4].

One of the very simple examples of shear flow with variable vorticity is a two-layer fluid with a free surface, in the upper layer of which there is a linear shear current and the lower layer is at rest. The study of the dispersion properties of wave motion for an infinitely deep lower layer was carried out in [5], and for a fluid of finite depth, it was performed in [6]. It was shown that the flow under consideration becomes unstable for a certain range of wavenumbers at sufficiently high shear flow velocity on the free surface [7].

Here, we describe the wave motion caused by the switching-on of an oscillating dipole (horizontal or vertical) located in the layer of fluid that is initially at rest. A similar problem has been recently solved for an oscillating source in [8].

2 MATHEMATICAL FORMULATION

We consider the horizontal layer of a homogeneous inviscid incompressible fluid of constant depth H bounded from above by a free surface and bounded from below by a horizontal bottom. In the unperturbed state, the part of fluid is at rest, and in the upper or lower layer of thickness h, there is a shear flow with a linear velocity profile (Figure 1). In the first case (a), the horizontal velocity in the upper layer is equal to $U(y) = U_0 y/h$, and in the second case (b), in the lower layer $U(y) = -U_0(y + H_1)/h$. The system of the Cartesian coordinates x, y is introduced so that, in the first case, the horizontal axis x coincides with the unperturbed interface between the shear and rest layers, and, in the second case, with the unperturbed free surface, the y axis is directed vertically upwards. The thickness of the layer at rest is equal to H_1, and the total depth of fluid is equal to $H = H_1 + h$.

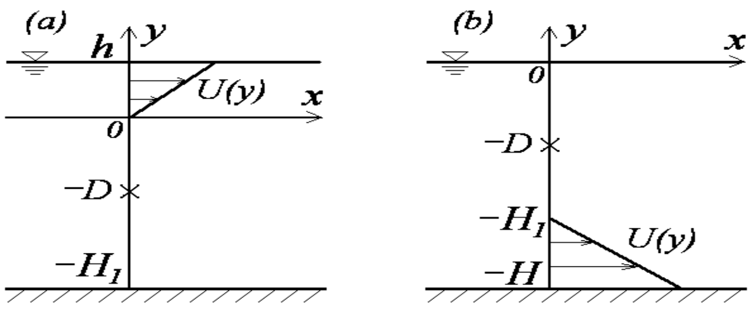

Figure 1. Schematic diagram.

It is assumed that in the fluid at rest either the horizontal or vertical dipole begins to operate at time $t = 0$ at point $x = 0,\ y = -D,\ 0 < D < H_1$, oscillating with angular frequency Ω.

2.1 Upper Shear Layer

In a shear layer, the linearized Euler equations have the form

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{v} + v\frac{d\mathbf{V}}{dy} + \frac{\nabla p_1}{\rho} = 0,\ \text{div}\ \mathbf{v} = 0 \quad (|x| < \infty,\ 0 \le y \le h), \tag{1}$$

where $\mathbf{V} = (U(y), 0)$ is the velocity vector of the main flow, $U(y) = U_0 y/h$, $\mathbf{v} = (u, v)$ are the velocity vector of disturbed motion, p_1 is the dynamic pressure, and ρ is the fluid density.

In the presence of the linear shear of the main flow, the wave motion components can be represented in the form $u(x, y, t) = \partial\phi_1/\partial x,\ v(x, y, t) = \partial\phi_1/\partial y$, where the function $\phi_1(x, y, t)$ satisfies the Laplace equation

$$\partial^2\phi_1/\partial x^2 + \partial^2\phi_1/\partial y^2 = 0 \quad (|x| < \infty,\ 0 \le y \le h).$$

The kinematic and dynamic conditions on the free surface of the fluid take the form

$$\partial\eta/\partial t + U_0\partial\eta/\partial x = v, \quad \rho g\eta = p_1 \quad (y = h),$$

where $\eta(x, t)$ is the surface elevation and g is the gravitational acceleration.

In the lower layer, the velocity potential $\Phi(x, y, t)$ can be presented as

$$\Phi(x, y, t) = \Phi_0(x, y, t) + \phi_2(x, y, t) \quad (|x| < \infty,\ -H_1 \le y \le 0), \tag{2}$$

where $\Phi_0(x, y, t)$ is the dipole velocity potential: for the horizontal dipole,

$$\Phi_0(x, y, t) = \frac{\mu_0}{2\pi}\frac{x\sin(\Omega t)}{[x^2 + (y + D)^2]}, \tag{3}$$

for the vertical dipole,

$$\Phi_0(x, y, t) = \frac{\mu_0}{2\pi}\frac{(y + D)\sin(\Omega t)}{[x^2 + (y + D)^2]}, \tag{4}$$

μ_0 is the dipole moment amplitude. The function $\phi_2(x, y, t)$ satisfies the Laplace equation.

At the interface between the upper and lower layers, the conditions of continuity of the vertical velocity and the pressure should be satisfied

$$\partial\phi_1/\partial y = \partial\Phi/\partial y,\ p_1 = -\rho\partial\Phi/\partial t \quad (y = 0),$$

The bottom condition is $\partial\Phi/\partial y = 0$ $(y = -H_1)$. At the initial instant of time, there are no wave disturbances

$$\phi_1 = \phi_2 = 0, \quad \eta = 0 \quad (t = 0). \tag{5}$$

2.2 Near-Bottom Shear Layer

In the upper layer of fluid, we seek the velocity potential $\Psi(x, y, t)$ in the form similar to (2)

$$\Psi(x, y, t) = \Phi_0(x, y, t) + \psi_1(x, y, t) \quad (|x| < \infty,\ -H_1 \le y \le 0),$$

where the function $\Phi_0(x, y, t)$ is given in (3) and (4) for horizontal or vertical dipole, respectively, and the function $\psi_1(x, y, t)$ satisfies the Laplace equation.

In the lower layer $(-H \le y \le -H_1)$, in which the shear flow $U(y) = -U_0(y + H_1)/h$ takes place, the linearized Euler equations similar to (1) are satisfied, and we seek the velocity components of wave motion in the form: $u(x, y, t) = \partial\psi_2/\partial x,\ v(x, y, t) = \partial\psi_2/\partial y$, where the function $\psi_2(x, y, t)$ satisfies the Laplace equation. The boundary conditions on the free surface are given by

$$\partial\eta/\partial t = \partial\Psi/\partial y, \quad g\eta + \partial\Psi/\partial t = 0 \quad (y = 0).$$

At the interface between the layers, we have

$$\partial\Psi/\partial y = \partial\psi_2/\partial y,\ \partial^2\Psi/\partial x\partial t = \partial^2\psi_2/\partial x\partial t - U_0/h\partial\psi_2/\partial y \quad (y = -H_1),$$

and on the bottom $\partial\psi_2/\partial y = 0$ $(y = -H)$. The initial conditions are similar to (5).

3 SOLUTION METHOD

To solve the initial-boundary-value problems formulated in Section 2, we use the Fourier and Laplace transforms in the following form:

$$\bar{\phi}_1(k, y, s) = \int_0^\infty e^{-st}\int_{-\infty}^\infty \phi_1(x, y, t)e^{-ikx}dxdt.$$

Similar transformations are introduced for the remaining unknown functions.

Using expansions in hyperbolic functions for $\bar{\phi}_{1,2}$ (for more details, see [8]), we obtain the solution for the function $\bar{\eta}(k,s)$ using the example of a vertical dipole for a fluid with an upper shear layer

$$\bar{\eta} = \mu_0 \Omega |k| \frac{[e^{-2|k|(H_1-D)} - 1]e^{-|k|(D+h)}}{1+e^{-2|k|H}} \frac{s(s+ikU_0)}{(s^2+\Omega^2)P(k,s)}.$$

Here, $P(k,s)$ is a third-degree polynomial

$$P(k,s) = s^3 + ia_1 s^2 + a_2 s + ia_3,$$

$$a_1(k) = 2kU_0 + \gamma[b_+ - \tanh(|k|H)],\ a_2(k) = g|k|\tanh(|k|H) + \gamma[kU_0(\tanh(|k|H) - 2b_+) + \gamma b_-] - k^2U_0^2, \tag{6}$$

$$a_3(k) = \gamma[kU_0(\gamma b_- - kU_0 b_+) + g|k|b_-],\ b_\pm(k) = (1 \pm e^{-2|k|h})\frac{1-e^{-2|k|H_1}}{2(1+e^{-2|k|H})},\ \gamma = \frac{U_0}{h}\mathrm{sgn}k. \tag{7}$$

The polynomial can be represented in the form $P(k,s) = \prod_{n=1}^{3}(s-s_n)$, where $s_n(k)$ $(n=\overline{1,3})$ are the roots of the equation $P(k,s)=0$. After performing the inverse Laplace and Fourier transforms, we obtain the following solution for the surface elevation:

$$\eta(x,t) = \frac{\mu_0\Omega}{\pi}\int_0^\infty kF(k)[A(k,t)\cos kx - B(k,t)\sin kx]dk, \tag{8}$$

$$F(k) = \frac{[e^{-2k(H_1-D)} - 1]e^{-k(D+h)}}{1+e^{-2kH}},\ A(k,t) + iB(k,t) = \sum_{n=1}^{5}\alpha_n(k)e^{s_n(k)t},\ s_{4,5} = \pm i\Omega.$$

The functions $\alpha_n(k)$ $(n=\overline{1,5})$ satisfy the equality

$$\frac{s(s+ikU_0)}{(s^2+\Omega^2)P_1(k,s)} = \sum_{n=1}^{5}\frac{\alpha_n(k)}{s-s_n(k)},$$

and their determination reduces to solving the system of five linear algebraic equations. In deriving (8), we used the property of the functions $s_n(k)$ $(n=\overline{1,3})$ and $\alpha_n(k)$ $(n=\overline{1,5})$, which means that their values for $k>0$ and $k<0$ are complex conjugate.

4 DISPERSION RELATIONS

It is possible to investigate the dispersion properties of the waves that arise in the cases under consideration. For each of the waves, the dispersion relation establishes the dependence of its frequency ω on the wavenumber k. Using the equation $P(k,s)=0$ and introducing the change of variable $\omega = is$, we obtain the polynomial for determining the dispersion relations for each of the three wave modes

$$\omega^3 - a_1(k)\omega^2 - a_2(k)\omega + a_3(k) = 0, \tag{9}$$

where the values of $a_n(k)$ $(n=\overline{1,3})$ are given in (6),(7). It is easy to see that for each of the three waves the equality $\omega_n(k) = -\omega_n(-k)$ is fulfilled.

For near-bottom shear layer, the dispersion relations are defined as the roots of a polynomial

$$\omega^3 - b_1(k)\omega^2 - b_2(k)\omega + b_3(k) = 0, \tag{10}$$

$$b_1(k) = \gamma f_+,\ b_2(k) = g|k|\tanh(|k|H),\ b_3(k) = g\gamma|k|f_-,\ f_\pm(k) = (1 \pm e^{-2|k|H_1})\frac{1-e^{-2|k|h}}{2(1+e^{-2|k|H})}.$$

5 NUMERICAL RESULTS

We use the following input parameters: $h = 10\ m,\ H_1 = 30\ m, H = 40\ m$. In Figure 1(a), the dispersion dependencies for the case of the upper shear layer are presented, defined as the roots of Eq. (9). Curves 1 and 2 correspond to a fluid at rest in an unperturbed state, for which there are only two wave modes $\omega = \pm\sqrt{gk\tanh kH}$. Curves 3-5 and 6-8 show the dispersion relationships for $U_0/\sqrt{gh} = 0.2,\ 0.5$, respectively. For given parameters, the wave motion is stable when U_0 is less then 7 m/s. Figure 2(b) shows the dispersion dependencies for the near-bottom shear layer, which are defined as the roots of Eq. (10), at $U_0/\sqrt{gh} = 0,\ 2.5,\ 5$.

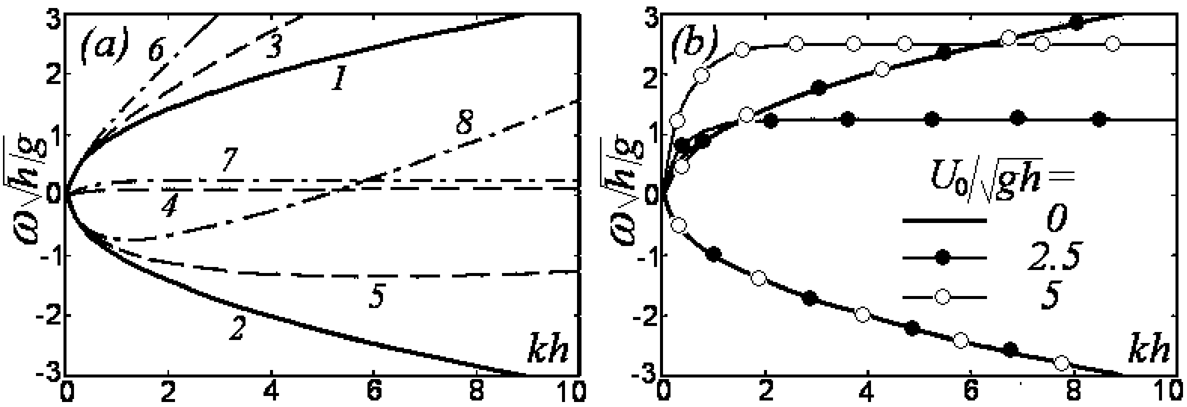

Figure 2. Dispersion dependencies.

It can be seen that in the presence of the near-bottom shear flow, two wave modes practically coincide with the case of a shear-free flow, and the third mode increases sharply with velocity U_0. In this case, no unstable wave motions were detected.

The free surface elevation is presented in Figure 3(a),(b) for the upper and the lower shear layers, respectively, at a fixed time value. In Figure 3(a), the values $U_0/\sqrt{gh} = 0.5,\ \Omega\sqrt{h/g} = 1,\ D = 10\ m,\ t = 150\ s$ were taken. In Figure 3(b), the values $U_0/\sqrt{gh} = 2.5,\ \Omega\sqrt{h/g} = 1.2,\ D = 20\ m,\ t = 300\ s$ were used.

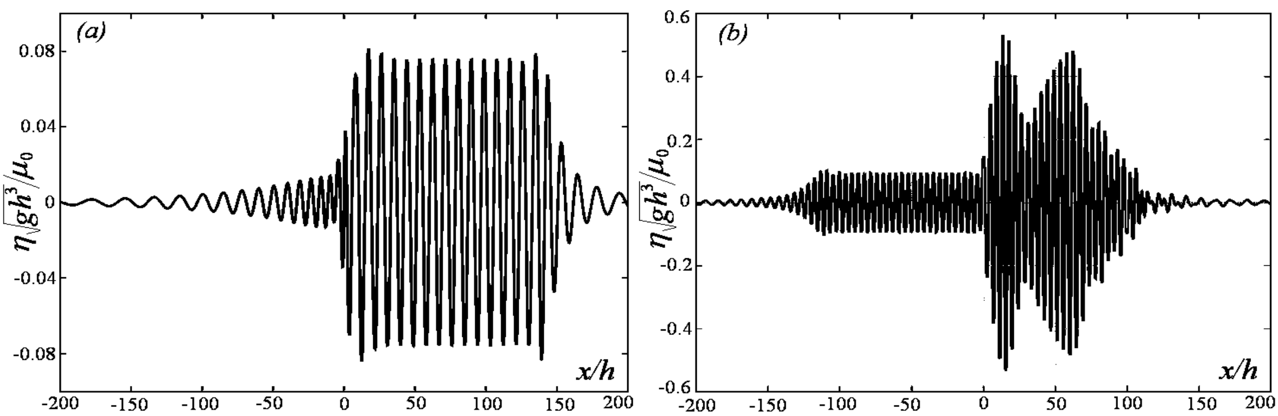

Figure 3. Free surface elevation.

More detailed numerical results will be presented at the Workshop.

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