Capillary waves generated by free surface detachment

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HIGHLIGHTS
We employ the Brillouin–Villat criterion to determine the position of the free-surface detachment from an arbitrary shaped body. The fluid is assumed to be inviscid and incompressible. It is shown that the surface tension generates capillary waves that closely resemble those calculated by Crapper [1].

1. Introduction

The separating flows of an inviscid liquid, accounting for surface tension, can be considered as a model of real flows at high Reynolds numbers, where wettability effects and the viscosity of the liquid are negligible. Such scenarios may arise in fluid-structure interactions at high speeds, including impact flows, cavity flows, and gliding flows. In these cases, a body interacts with a free surface flow that detaches at a specific point on the body. The theoretical study of the effect of surface tension effect on flow detachment has a long history starting from the works of Rayleigh and Kelvin (see [2], p. 455) with application to linear theory of water waves, and the work of Joukowskii (see [3], p. 549) who included surface-tension in the Bernoulli equation and derived a nonlinear boundary condition along the free streamline. He obtained a fully nonlinear solution for the flow past a bubble between two parallel walls. Later McLeod [4] considered the bubble problem in an infinite space. Crapper [1] found an exact solution for capillary waves propagating at the surface of an irrotational flow of infinite depth. Even though the classical Helmholtz–Kirchhoff solution for free streamlines past a flat plate was obtained over a century ago, the flow detachment mechanism in the presence of surface tension remains poorly understood. The challenge arises from the infinite curvature predicted by potential flow theory at the sharp trailing edge, where detachment occurs. Even a tiny amount of surface tension leads to infinite pressure and velocity in the Bernoulli equation. The infinite pressure implies that the flow remains attached around the sharp edge. In our work, we study the effect of surface tension on the angle of the free-surface detachment, and the shape of the free surface, in the case of Kirchhoff flow past a circular cylinder and a flat plate with rounded edges. We apply the integral hodograph method to derive the complex velocity potential and reduced the problem to a system of nonlinear equations with respect to the velocity magnitude on the free surface and solve it numerically using a collocation method.

2. Formulation of the problem

We study a two-dimensional irrotational free-streamline flow of an inviscid, incompressible fluid past a circular cylinder, as sketched in Figure 1a. Gravity is neglected, but surface tension is considered. The radius of the cylinder is denoted by R and we introduce a Cartesian coordinate system XY, with the origin at the center of the cylinder. The flow is symmetric with respect to the X −axis, which is aligned with the inflow velocity U. Flow separation occurs at point O on the cylinder, forming a free-streamline OTB, which extends to infinity. Surface tension may generate waves on the free surface, extending downstream to infinity. To satisfy the radiation condition, we apply the dynamic boundary condition on the segment OT of the free-streamline. On the remaining part of the streamline, TB, the velocity magnitude remains constant and equal to the inflow velocity U.

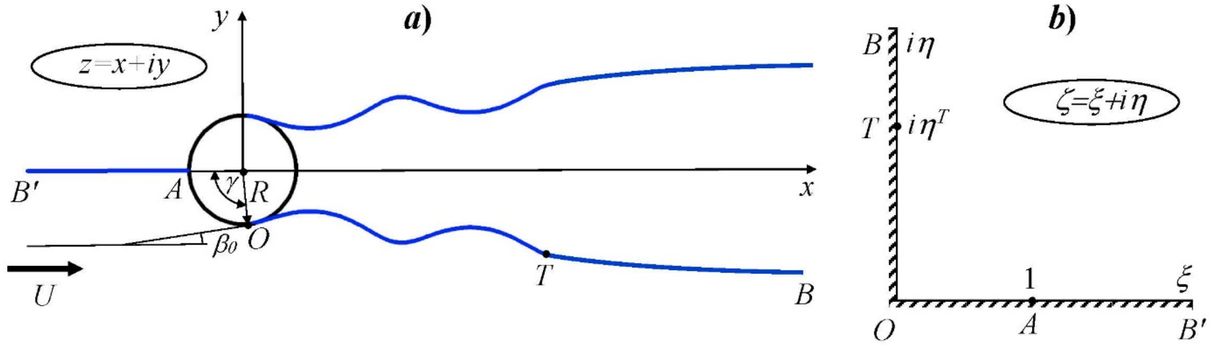

Figure 1. (a) Sketch of the free-streamline flow past a circular cylinder, and (b) the parameter plane.

We introduce the complex velocity potential, $W(Z) = \Phi(X,Y) + i\Psi(X,Y)$, where $\Phi(X,Y)$ is the velocity potential and $\Psi(X,Y)$ is the streamfunction, and $Z = X + iY$. The boundary-value problem for the velocity potential can be formulated as follows:

$$\nabla^2\Phi = 0, \qquad \nabla^2\Psi = 0 \tag{1}$$

in the liquid domain;

$$\frac{\partial\Phi}{\partial Y} = \frac{\partial\Phi}{\partial X}\frac{dY_b}{dX}, \qquad \Psi = 0 \tag{2}$$

on the body surface $Y_b(X)$ including AB'; and

$$\rho\frac{V^2}{2} + p = \rho\frac{U^2}{2} + p_\infty, \qquad \Psi = 0, \qquad 0 < X < X_T, \quad Y = Y(X), \tag{3}$$

which is the dynamic boundary condition at the free-streamline, $Y = Y(X)$. Here, $V = |\nabla\Phi|$ is the velocity magnitude, $p(X)$ is the hydrodynamic pressure on the liquid side of the free-streamline and p_∞ is its value at infinity; ρ is the density of the liquid;

$$p = p_\infty, \qquad \Psi = 0, \qquad X_T < X < \infty, \quad Y = Y(X), \tag{4}$$

which is the dynamic boundary condition on the remainder of the free-streamline;
and the far-field condition

$$\nabla\Phi \to U, \qquad |X^2 + Y^2| \to \infty. \tag{5}$$

To complete the formulation of the boundary-value problem (1) – (5), an equation for the hydrodynamic pressure at the free-streamline is needed. The surface tension affects the pressure jump across the free-streamline according to the Laplace-Young condition:

$$p - p_c = \tau\kappa, \tag{6}$$

where p_c is the pressure inside the cavity, τ is the coefficient of surface tension and κ is the curvature of the free-streamline. In Kirchhoff-type flows with an infinite cavity, the curvature of the free streamlines approaches zero at infinity; therefore, the pressure within the cavity is the same as the pressure at infinity, $p_c = p_\infty$. Using the integral hodograph method, which is based on the integral formula for solving mixed boundary value problems for complex functions, the expression for the complex velocity is derived:

$$\frac{dw}{dz}=v_0\sqrt{\frac{1-\zeta}{1+\zeta}}\exp\left[\frac{1}{\pi}\int_0^1\frac{d\delta_b}{d\xi}\ln\left(\frac{\xi-\zeta}{\xi+\zeta}\right)d\xi-\frac{i}{\pi}\int_0^\infty\frac{d\ln v}{d\eta}\ln\left(\frac{i\eta-\zeta}{i\eta+\zeta}\right)d\eta\right], \tag{7}$$

where $\delta_b(\xi)$ and $v(\eta)$ are the functions of the slope of the body and velocity magnitude on the free surface, respectively. The derivative of the complex potential, derived through the conformal mapping of the first quadrant onto a strip in the ww-plane, is expressed as:

$$\frac{dw}{d\zeta}=K\zeta, \tag{8}$$

where K being a real constant.

Applying the kinematic and dynamic boundary conditions yields the following integral equations, expressed in terms of the functions $\delta_b(\xi)$ and $v(\eta)$:

$$\frac{d\delta_b}{d\xi}=\frac{d\delta_b}{ds_b}\frac{K\xi}{v_0}\sqrt{\left|\frac{1+\xi}{1-\xi}\right|}\exp\left[-\frac{1}{\pi}\int_0^1\frac{d\delta_b}{d\xi'}\ln\left|\frac{\xi'-\xi}{\xi'+\xi}\right|d\xi'-\frac{1}{\pi}\int_0^\infty\frac{d\ln v}{d\eta}\left(\pi-2\tan^{-1}\frac{\eta}{\xi}\right)d\eta\right]. \tag{9}$$

$$\frac{1}{\eta}\left(\frac{2}{\pi}\int_0^\infty\frac{d\ln v}{d\eta'}\frac{\eta' d\eta'}{\eta'^2-\eta^2}-\frac{1}{1+\eta^2}-\frac{2}{\pi}\int_0^1\frac{d\delta_b}{d\xi}\frac{\xi d\xi}{\xi^2+\eta^2}\right)=\frac{We\,K}{2v(\eta)}(1-v^2). \tag{10}$$

The Brillouin-Villat criterion [3,5] was developed for free-streamline separation without considering surface tension. However, since the criterion has a geometric nature – stating that the free-streamline should not cross the body – it can also be applied to determine the free-streamline separation in the presence of non-zero surface tension. Specifically, this means that

$$\lim_{s_b\to 0}\frac{d\ln v_b}{ds_b}=\lim_{\xi\to 0}\frac{d\ln v_b}{d\xi}\Big/\frac{ds}{d\xi}=0,$$

where s_b is the arc length coordinate on the body. By taking the derivative of the magnitude of the complex velocity (7) with $\zeta=\xi$ and differentiating it with respect to ξ, the above equation is obtained

$$\int_0^1\frac{d\delta_b}{d\xi}\frac{d\xi}{\xi}-\int_0^\infty\frac{d\ln v}{d\eta}\frac{d\eta}{\eta}+\frac{\pi}{2}=0. \tag{11}$$

3. Results. The lower part of the flow past a circular cylinder is shown in figure 2 for various We numbers, including the case for $We=\infty$, corresponding to zero surface tension. The predicted detachment angle for this case is $\gamma=55.03°$, which matches the known value [2]. For $We=\infty$, the slope of the free surface decreases monotonically from γ at the detachment point to zero at infinity. In this case, the velocity magnitude along the entire free surface, including the detachment point, remains constant, $v(\eta)\equiv v_\infty=1$.

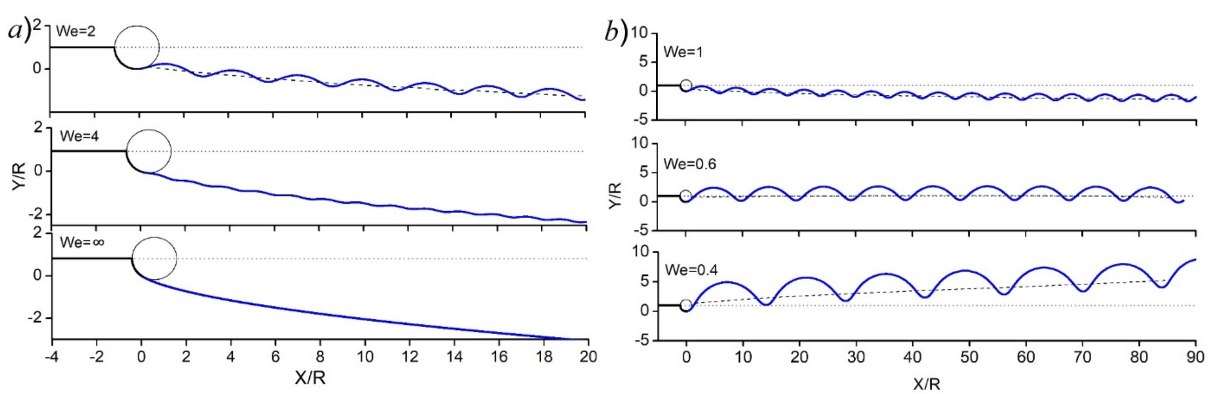

Figure 2. Free surface shapes for various Weber numbers ($We=\rho U^2/\tau$).

When $We < \infty$, the free surface exhibits capillary waves, with both the wavelength and amplitude increasing as surface tension increases. The truncation region TO, where the velocity remains constant and equals the velocity at infinity, is not shown in Figure 2. In the right-hand figure, a dashed line crosses the free surface at points where the velocity equals 1, while a dotted line represents the flow's axis of symmetry. It can be seen that the intervals below the dashed line, where the velocity magnitude $v > 1$, are shorter than those above the line, where $v < 1$. This characteristic of capillary waves was revealed by Crapper [1]. For $We = 1$, the slope of the dashed line starts off negative but gradually approaches zero at infinity. At $We = 0.6$, the slope is nearly zero from the outset. At $We = 0.4$, the slope of the dashed line becomes positive but also gradually tends to zero at infinity. For $We = 1$, the first wave crest almost touches the flow symmetry axis (dotted line). As We decreases to 0.6, the wavelength and amplitude increase further, causing the free surface to cross the symmetry line. This situation is physically impossible for flow around a cylinder because the upper and lower free streamlines would collide behind the cylinder. However, such a flow could still occur if it originated from a semi-infinite solid plate with a rounded trailing edge.

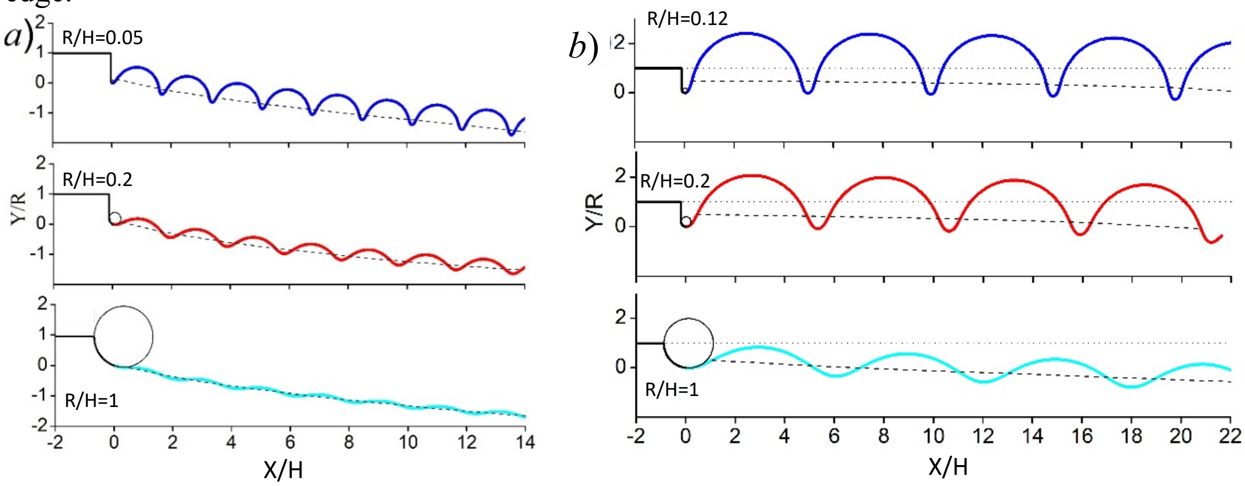

Figure 3. Flow past a flat plate with the rounded trailing edge for (*a*) $We = 3$, (*b*) $We = 1$.

The solution method capable to investigate the flow detachment from a flat plate with rounded edges. The width of the plate is given by $H = 2(L + R)$, where L is the length of the flat portion, and R is the radius of the rounded edges of the plate. The Weber number is based on the total width H instead of R. In the special case where the radius $R \rightarrow 0$, the configuration approaches to a flat plate with sharp edges. Figure 3 shows the free surface shape for various edge radii for (*a*) $We = 3$ and (*b*) $We = 1$. For different radii, the wavelength remains approximately constant, while the wave amplitude increases as the radius decreases, for both $We = 3$ and $We = 1$. Additionally, the wavelength increases as the Weber number decreases.

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