Experiment of modulational wave train under sea ice in Wave-Ice Tank and Numerical calculation by Nonlinear Schrödinger Equation with Raman Scattering Term

Koya Sato, Takuji Waseda, Tsubasa Kodaira
Graduate School of Frontier Sciences, The University of Tokyo, Chiba, Japan
sato-koya874@g.ecc.u-tokyo.ac.jp

HIGHLIGHTS

To explain the anomalous downshifting observed in our experiments, a Raman Scattering Term is introduced into the Nonlinear Schrödinger Equation (NLS). The Raman Scattering Term does not change the total energy of the waves but their momentum, providing a potential mechanism for the observed spectral downshifting.

1 Introduction

The spectral transition to the low-frequency side of the wave spectrum, known as spectral downshift, occurs in ocean due to nonlinear effects. In open water, through wind-wave growth and four-wave resonance, spectral downshift is ubiquitous (Phillips, 1958) [1]. Sea ice exponentially dissipates waves with frequency-dependent attenuation properties $\exp(-Kf^n x)$, causing a downshift because low-frequency waves survive longer. However, the observational results of waves propagating under sea ice in the Okhotsk Sea by Waseda et al. (2022) [2] revealed anomalous spectral downshift that cannot be explained by wave attenuation alone, as the low-frequency component grows. Currently, no theory exists to explain this phenomenon.

To confirm the existence of nonlinear energy exchange in waves propagating under sea ice and unravel its mechanism, we conducted both tank experiments and numerical calculations.

2 Modulation Instability and Numerical Calculation

2.1 Modulation instability (MI)

Benjamin & Feir (1967) [3] discovered that Stokes waves are unstable to the growth of sideband waves (sideband waves; are two wave components with frequencies slightly higher/lower than the carrier wave frequency). The initial growth rate β of the modulational wave train, when we introduce $\hat{\delta} = \delta/ak$, is given by:

$$\beta = \frac{d\big(ln(a)\big)}{kdx} = (ak)^2\,\hat{\delta}\sqrt{2-\,\hat{\delta}^2}$$

2.2 Nonlinear Schrödinger Equation (NLS)

Expressing the time evolution of surface elevation by using the complex amplitude $A(x,y,t)$ of the water surface as $Re[A(x,y,t)\exp(i(kx-\omega t)]$, the temporal evolution of A, under the assumption of a narrow frequency bandwidth $O(\Delta\omega/\omega) = O(ak)$, is given by the following Nonlinear Schrödinger Equation (NLS):

$$i\frac{\partial A}{\partial t} - \frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2} - \frac{\omega_0^2 k_0^2}{2}|A|^2 A = 0$$

If we solve this equation numerically and calculate spectrum, due to the symmetricity of terms of NLS, symmetric growth will be observed for the upper and lower sidebands $f_0 \pm \Delta f$(and $f_0 \pm 2\Delta f$). Alberto et al. (2023) [4] modeled spectral downshifting under sea ice through numerical simulations using a dissipative nonlinear Schrödinger equation (d-NLS) with a frequency-dependent dissipation term added to the NLS.

3 Experiment in Wave-Ice Tank

3.1 Experimental Methods

The experimental facility is in Kashiwa Campus, the University of Tokyo, and was constructed by the Japan Agency for Marine-Earth Science and Technology (JAMSTEC). Its name is Wave-Ice Tank. To measure the generated waves, four gauges were used and ice fences to protect sensing wires in the wave height gauges were installed.

Ice was created in the tank, and a modulational wave train (wave train subject to modulational instability), consisting of a carrier wave and upper/lower sidebands. In the process of creating ice in the tank, the room temperature was lowered for freezing while wave generation was performed simultaneously. We call this procedure "*wave-induced ice formation*".

Through this procedure, ice particles, depicted in Figure 2, are generated in the tank. In terms of actual sea ice classification, this ice is akin to an "ice rind", and upon completion of the ice generation process, the ice particles uniformly cover the surface of the tank.

Figure 2. Generated ice in Wave-Ice Tank

3.2 Experimental Results

The spectrum and energy of each peak for the modulational wave train are illustrated in Fig. 3.

This wave train shows temporary sideband growth due to modulation instability when the length of the tank is sufficiently large. However, since our tank was only 8 meters long, no spectral changes due to modulation instability were observed.

In our experiment, the amplitude of waves is attenuated due to the presence of ice. Figure 3(a) shows the power spectrum plotted at each gauge position. The shape of the spectrum changes significantly between the third gauge and the fourth gauge. To determine whether this spectral change can be explained solely by attenuation caused by the presence of ice or not, we plot the energy of each wave component individually as Figure 3(b).

In this figure, we can observe that the component $f - 2\Delta f$ is growing. Therefore, it is evident that through nonlinear energy transfer, energy from the carrier wave (or upper sideband) is being transferred into waves with frequencies lower than the carrier wave. This is clear evidence of nonlinear energy transfer occurring while waves propagate under ice.

However, the issue at hand is why nonlinear energy exchange is occurring. Modulation instability is known to result in spectral downshift on the scale of several tens of waves, but the experimental results indicate downshifting in this experiment is on the scale of about one wave.

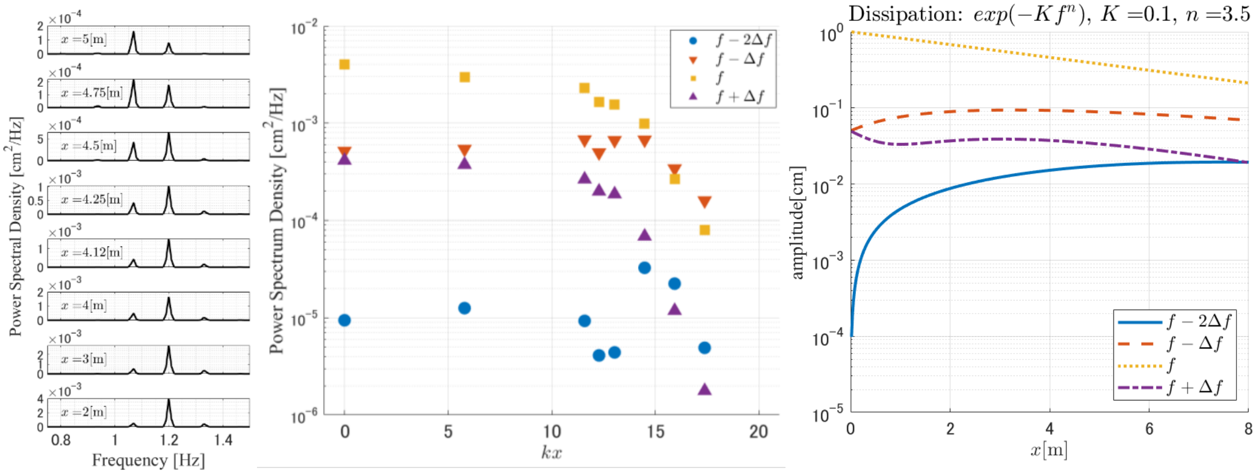

Figure 3. Experimental result of modulational wave train, (a)powerspectrum (b)energy change in each wave. This modulational wave train consists of lower sideband $f-\Delta f = 1.07Hz$, carrier wave $f = 1.2Hz$ and upper sideband $f+\Delta f = 1.33Hz$. (c) is the numerical calculation by NLS+dissipation+Raman Scattering term

3.3 Numerical Calculation by NLS with dissipation

To check if the downshifting in our experiments is from the evolution of waves by modulational instability and dissipation that higher frequency component selectively vanishes, we conducted numerical calculations by using NLS which dissipation is implemented. The method to solve NLS is based on Lo&Mei (1985) [5] However, we cannot see the evolution of $f-2\Delta f$ component around the center of the tank. Therefore, modulational instability and dissipation cannot explain downshifting in this experiment.

4 Raman Scattering Term

4.1 Introduction of Raman Scattering Term and Comparison to the Experimental Results

In the field of optics, NLS is used to calculate the spectral evolution of lights in fibers, and the mathematical properties of this term are also discussed [6]. One of the major differences between NLS in oceanography and optics is the existence of Raman Scattering Term which causes spectral upshifting and downshifting.

In our experimental results, very fast spectral downshifting occurs, and previous studies cannot explain this fast downshifting. Therefore, we must implement something new to NLS to understand the mechanism behind the experimental results. We artificially added Raman Scattering Term to NLS like below.

$$i\left(\frac{\partial A}{\partial t}+c_g\frac{\partial A}{\partial x}\right)-\frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2}-\frac{\omega_0 k_0^2}{2}|A|^2A-\boldsymbol{\Gamma A\frac{\partial}{\partial x}|A|^2}=0$$

We calculated NLS with Raman Scattering Term and dissipation as shown in Figure 3(c).

In this calculation, the spectral evolution of $f-2\Delta f$ is enhanced. Therefore, spectral downshifting might be explained by adding the Raman Scattering Term to the NLS.

4.2 Physical Meaning of Raman Scattering Term in Ocean Wave

We calculated the spectral evolution of the modulational wave train by using NLS with only Raman scattering term but without dissipation to investigate how the Raman scattering term affects the energy and momentum of waves.

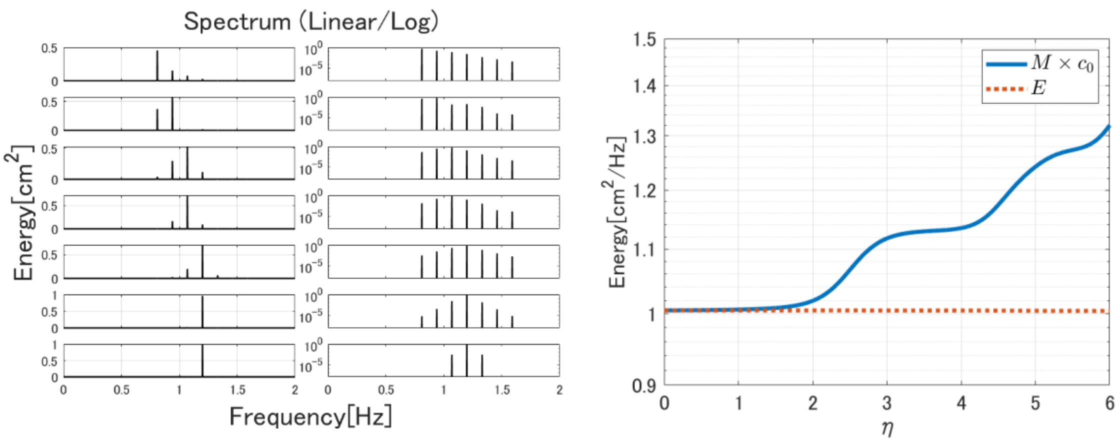

Figure 4. Numerical calculation of NLS+Raman Scattering Term, (a) Powerspectrum of waves (b) plot of E and $M \times c_0$, here E and M is total energy and momentum respectively, c_0 is the phase speed of the carrier wave.

In figure 4(a), wave energy continuously and permanently cascades to lower components. Therefore, Raman scattering term causes permanent downshifting to wave spectrum.

But why does the Raman scattering term cause spectral downshifting? The answer lies in the change in momentum. Tulin & Waseda (1999) [5] showed that spectral downshifting during wave breaking occurs due to an imbalance between energy loss and momentum loss as equation below.

$$\frac{\partial}{\partial t}(E_{-1} - E_{+1}) = -\frac{D_{dis} - c_0 \dot{M}_{dis}}{\Delta f / f}$$

Here, D_{dis} and M_{dis} is energy and momentum loss. We have confirmed that in calculations of the standard NLS without the Raman scattering term, energy and momentum take almost the same values. The addition of the Raman Scattering Term corresponds to momentum change, as shown in the figure 4(b). The Raman scattering term does not change the total energy of the waves but does change their momentum. Therefore, based on the equation from Tulin & Waseda (1999)[5], it can be understood that the Raman scattering term, which increases momentum, contributes to spectral downshifting.

Energy attenuation caused by sea ice can be applied to the NLS through the implementation of a damping term, and momentum changes can be applied through the implementation of the Raman Scattering Term. Thus, if the fast spectral downshifting due to the presence of sea ice is a phenomenon caused by energy and momentum loss, similar to wave breaking, it may be possible to reproduce our experimental results using the NLS with the Raman Scattering Term.

REFERENCES

[1] Phillips, O. M. (1958). Journal of Fluid Mechanics, 4(4), 426-434.
[2] Waseda, T., Alberello, A., Nose, T., Toyota, T., Kodaira, T., and Y. Fujiwara. 2022. Philos. Trans. Royal Soc. A. 380.
[3] Benjamin, T. B., and Feir, J. B. 1967. Journal of Fluid Mechanics. vol. 27. 417–430
[4] Alberello, A., Părău, E. and Chabchoub, A. 2023. *Sci Rep* **13**, 13654.
[5] Tulin, M. P., and Waseda, T. 1999. Journal of Fluid Mechanics. 378. 197–232.
[6] Lo, E., & Mei, C. C. 1985. Journal of Fluid Mechanics, 150, 395-416.
[7] Kishimoto, N., Tsutsumi, Y. 2020. Mathematics of Wave Phenomena. Trends in Mathematics. Birkhäuser, Cham.