Homogenized Boussinesq and KdV models for anisotropic propagation of water waves over a structured ridge

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1. Introduction

The propagation of gravity waves over variable bathymetry has been extensively studied over the past 40 years [1]. In the linear regime, several theoretical, numerical, and experimental studies have demonstrated the anisotropic nature of effective wave propagation over rapidly varying periodic bathymetries in the long-wavelength/shallow-water regime [2, 3]. More recently, this topic has gained attention in the context of metamaterials for coastal protection [4, 5, 6]. However, solitons -which can cause significant damage to offshore structures and coasts (e.g., rogue waves) - are inherently nonlinear, and their interaction with variable bathymetry remains poorly understood. This study revisits the problem of nonlinear wave motion on the free surface of a liquid column with a periodically varying bottom [7, 8, 9]. We first derive the corresponding anisotropic homogenized Boussinesq and KdV equations and then analyze how soliton properties, such as celerity and spatial extent, are modified by the presence of rapidly varying bathymetry.

2. Non-linear gravity wave propagation over a periodic bathymetry

We consider the nonlinear propagation of water waves over a structured ridge with periodicity ph_0, resulting in water depths alternating between h_m and h_0, see figure 1. Assuming an inviscid, incompressible fluid in irrotational motion, the velocity $\boldsymbol{u}$ and its associated velocity potential φ satisfy the Laplace equation

$$\mathrm{div}^{3\mathrm{d}}\, \boldsymbol{u}(\boldsymbol{r}, z, t) = 0, \quad \boldsymbol{u}(\boldsymbol{r}, z, t) = \boldsymbol{\nabla}^{3\mathrm{d}}\varphi(\boldsymbol{r}, z, t), \tag{1}$$

where t is time, and $\boldsymbol{r} = (x, y)$ denotes the horizontal coordinates. The equations above are complemented by the dynamic and kinematic boundary conditions at the free surface, *i.e.*

$$\frac{\partial \varphi}{\partial t} + \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2} + g\eta = 0, \qquad u_z = \frac{\partial \eta}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\eta \quad \text{at } z = \eta(\boldsymbol{r}, t), \tag{2}$$

where g is the gravitational constant, and the gradient operators are defined as $\boldsymbol{\nabla} f = \partial_x f\, \mathbf{e}_x + \partial_y f \mathbf{e}_y$ and $\boldsymbol{\nabla}^{3\mathrm{d}} f = \boldsymbol{\nabla} f + \partial_z f \mathbf{e}_z$. A vanishing normal velocity condition, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$, is imposed on the rigid walls.

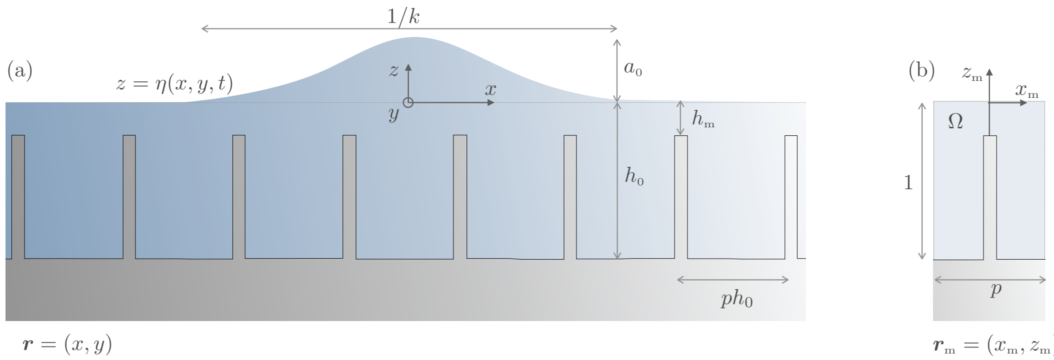

Figure 1: Three-dimensional propagation of nonlinear gravity waves with typical wavenumber k and amplitude a_0 over a sea bottom at depth h_0 supporting a rapidly varying bathymetry consisting of a periodic array of plates with height $h = h_0 - h_{\rm m}$ and periodicity ph_0.

3. Main results of the study

Using an asymptotic multi-scale approach up to the third order, we analyze the problem under (i) shallow-water, (ii) long-wavelength, and (iii) weakly nonlinear assumptions. This method reduces the full three-dimensional problem to a two-dimensional one at the free surface, employing a homogenization technique recently proposed for nonlinear water wave propagation over a step-like bathymetry [11].

Anisotropic effective Boussinesq equations. We derived effective equations govern the homogenized surface wave elevation $\eta(\boldsymbol{r}, t)$ and velocity at the free surface $\boldsymbol{u}(\boldsymbol{r}, t)$, averaged over the periodic cell. These quantities satisfy a Boussinesq-type equation of the form

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \operatorname{div}\left[(\alpha_x \overline{h} + n_x \eta) u_x \mathbf{e}_x + (\overline{h} + \eta) u_y \mathbf{e}_y\right] \\ \qquad + \overline{h} h_0^2 \operatorname{div}\left[\left(d_{xx}\dfrac{\partial^2 u_x}{\partial x^2} + d_{xy}\dfrac{\partial^2 u_x}{\partial y^2}\right)\mathbf{e}_x + \left(d_{yx}\dfrac{\partial^2 u_y}{\partial x^2} + d_{yy}\dfrac{\partial^2 u_y}{\partial y^2}\right)\mathbf{e}_y\right] = 0, \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + g\boldsymbol{\nabla}\eta + \dfrac{1}{2}\boldsymbol{\nabla}\left(n_x u_x^2 + u_y^2\right) = 0, \end{cases} \tag{3}$$

where $\overline{h}$ is the average water depth and $(\alpha_x, n_x, d_{xx}, d_{xy}, d_{yx}, d_{yy})$ are effective non-dimensional effective parameters characterizing nonlinear anisotropic propagation. For a *flat* bathymetry of constant depth h_0, these parameters reduce to $\overline{h} = h_0$, $\alpha_x = n_x = 1$, $d_{xx} = d_{xy} = d_{yx} = d_{yy} = 1/3$, resulting in the classical Boussinesq equation for flat bathymetry

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \operatorname{div}\left[(h_0 + \eta)\boldsymbol{u}\right] + \dfrac{h_0^3}{3}\operatorname{div}\left[\Delta \boldsymbol{u}\right] = 0, \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + g\boldsymbol{\nabla}\eta + \dfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) = 0, \end{cases} \tag{4}$$

The effective parameters are given by the resolution of linear static problems, of the Poisson type, which depend only on the geometry of the unit cell, as it is classical in asymptotic homogenization.

Anisotropic effective KdV equation and solitons. We now look at unidirectional wave propagation along $\mathbf{e}_X$ forming an angle θ with $\mathbf{e}_x$, with X being the position in this direction. The anisotropic KdV equation governing the propagation in this direction is given by

$$\frac{\partial \eta}{\partial t} + c_\theta \left(1 + \frac{3}{2}\frac{\eta}{h_\theta}\right)\frac{\partial \eta}{\partial X} + \gamma_\theta \frac{c_\theta h_\theta^2}{6}\frac{\partial^3 \eta}{\partial X^3} = 0, \quad c_\theta = \Gamma_\theta \sqrt{g\bar{h}}, \tag{5}$$

resulting in the closed-form equation of an effective soliton of amplitude η_0 and celerity u_θ given by

$$\eta(X,t) = \eta_0 \operatorname{sech}^2 \left\{ \left(\frac{3\eta_0}{4\gamma_\theta h_\theta^3}\right)^{1/2} (X - u_\theta t) \right\}, \quad u_\theta = c_\theta \left(1 + \frac{\eta_0}{2h_\theta}\right), \tag{6}$$

which depend on parameters that are given explicitly as functions of the effective coefficients

$$\begin{cases} h_\theta = \dfrac{\alpha_x \cos^2\theta + \sin^2\theta}{n_x \cos^2\theta + \sin^2\theta}\,\bar{h}, \qquad \Gamma_\theta^2 = \left(\alpha_x \cos^2\theta + \sin^2\theta\right), \\[2ex] \gamma_\theta = \dfrac{3}{\Gamma_\theta^2}\left(\dfrac{h_0}{h_\theta}\right)^2 \left(d_{xx}\cos^4\theta + (d_{xy} + d_{yx})\cos^2\theta \sin^2\theta + d_{yy}\sin^4\theta\right). \end{cases} \tag{7}$$

Figures 2 and 3 highlight the significant impact of variable bathymetry on the soliton shape: its thickness, or spatial extend, and velocity u_θ in (6). We considered a periodic bathymetry made of plates with *vanishing* thickness. By doing so $\bar{h} = h_0$ which leaves us with two non-dimensional geometrical parameters being the rescaled periodicity of the array p and smaller water depth $\xi = h_m/h_0$, due to the bathymetry ($\xi = 1$ corresponds to flat bathymetry). Both the spatial extent and velocity of solitons increase with increasing p or ξ. The anisotropic nature of solitons is particularly evident in their directional dependence ($\theta = 0$ along x across the plates and $\theta = \pi/2$ along y parallel to the plate in figure 3), emphasizing the role of structured bathymetry in wave dynamics.

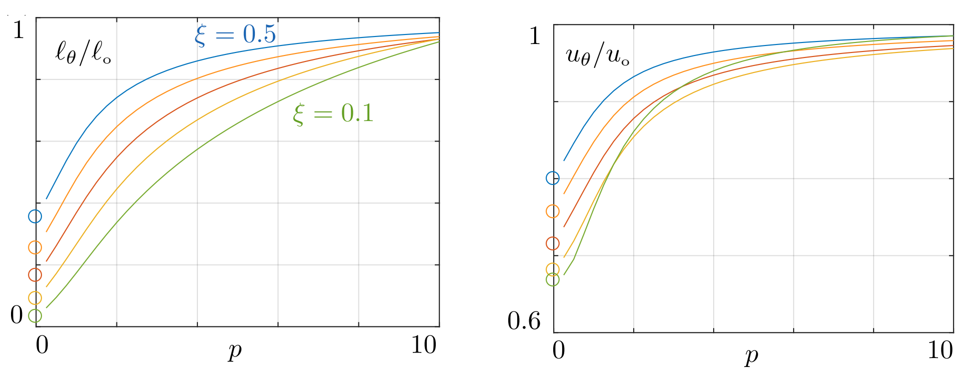

Figure 2: Influence of the periodicity p and minimum water depth ξ on the soliton thickness $\ell_\theta = \left(\frac{4\gamma_\theta h_\theta^3}{3\eta_0}\right)^{1/2}$ and velocity u_θ ($\theta = 0$, $h_0 = 10^{-2}$ m, $\eta_0 = 0.3h_0$, $g = 9.8$ m.s^{-2}).

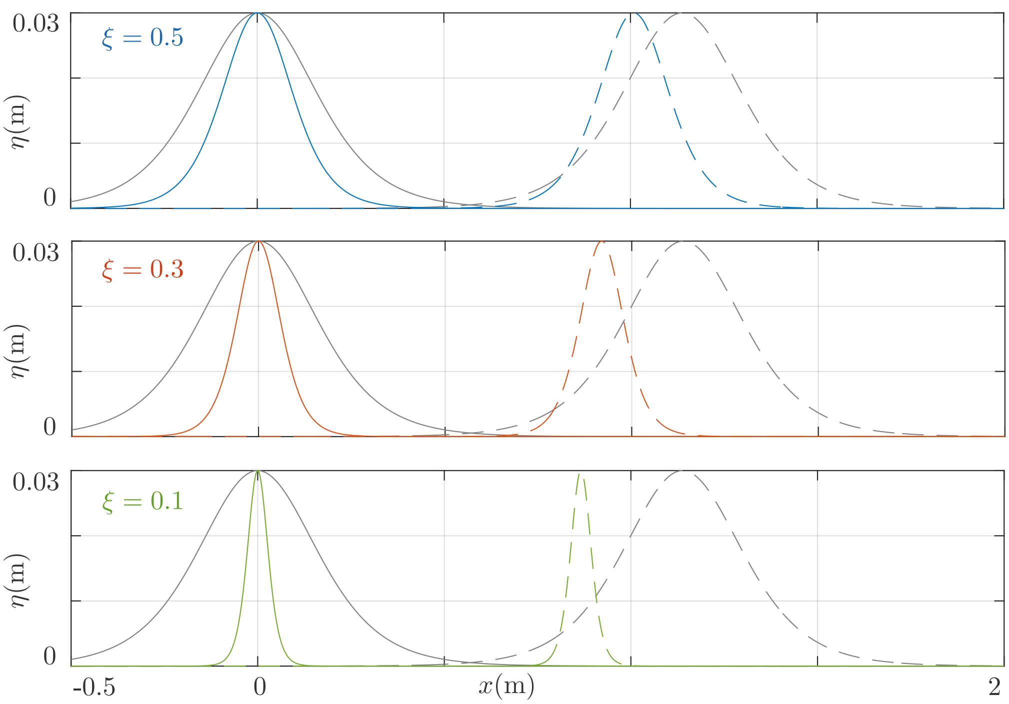

Figure 3: Soliton shapes $\eta(x, t)$ propagating along x ($\theta = 0$), at $t = 0$ (solid lines) and $t = 1$ s (dashed lines), from (6). The grey lines show the reference soliton shape for flat bathymetry and corresponding to propagation for $\theta = \pi/2$.

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