Six boundary-integral representations of the flow created by a ship that steadily advances in calm water

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Highlights: *The fundamental issue of defining a boundary-integral equation within a linear analysis of potential flow around a ship that steadily advances in calm water is examined. Six boundary-integral flow representations, associated with five alternative linear flow models, are given. These six alternative flow representations include a remarkably simple new boundary-integral equation that does not involve the flow potential at the waterline.*

The length and the speed of the ship are denoted as L and V, g is the acceleration of gravity, and $F \equiv V/\sqrt{gL}$ denotes the Froude number. The flow potentials at points $\boldsymbol{\xi} \equiv (\xi, \eta, \zeta \leq 0)$ and $\mathbf{x} \equiv (x, y, z \leq 0)$ associated with the Green function $G \equiv G(\boldsymbol{\xi}, \mathbf{x})$ that satisfies the Kelvin-Michell linear boundary condition $G_\zeta + F^2 G_{\xi\xi} = 0$ at the free-surface plane $\zeta = 0$ are denoted as $\varphi \equiv \varphi(\boldsymbol{\xi})$ and $\phi \equiv \varphi(\mathbf{x})$ hereafter. The ζ axis is vertical and points upward, and the ξ axis lies along the straight path of the ship and points toward the ship bow. The coordinates $\boldsymbol{\xi}$, $\mathbf{x}$ and the flow potential φ are nondimensional with respect to L and g.

1. The NK (Neumann-Kelvin) and NM (Neumann-Michell) flow representations

The first of the six alternative boundary-integral flow representations considered in this study is the classical **Neumann-Kelvin (NK) flow representation**, which is obtained via a straightforward application of Green's fundamental identity to the Green function G that satisfies the Kelvin-Michell linear free-surface boundary condition and the flow potential φ in the region bounded by the undisturbed free surface Σ^F outside the mean wetted ship-hull surface Σ^H. The NK flow representation is expressed in [1] in the *weakly-singular* form

$$\phi = \phi^H + F^2 \int_\Gamma d\eta \, [\, (\phi - \varphi) \, G_\xi + G \, \varphi_\xi \,] \quad \text{where} \tag{1}$$

$$\phi^H \equiv \phi^H(\mathbf{x}) \equiv \int_{\Sigma^H} da(\boldsymbol{\xi}) \Big[G \, q^H(\boldsymbol{\xi}) + (\phi - \varphi) \, \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} G \Big] \;\; \text{with} \;\; q^H \equiv \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} \varphi \;\; . \tag{2}$$

The unit vector $\mathbf{n} \equiv \mathbf{n}(\boldsymbol{\xi}) \equiv (n^x, n^y, n^z)$ in (2) is normal to the ship-hull surface Σ^H and points outside the ship. The ship-hull flux q^H in (2) is *presumed known* in the NK boundary-integral flow representation (1) and the alternative flow representations considered further on. In particular, the ship-hull flux q^H in (2) is given by $q^H = F \, n^x$ if the influence of the viscous boundary layer is ignored. Σ^H in the hull-surface integral (2) associated with the classical NK linear flow model represents the mean wetted ship-hull surface below the *undisturbed* free-surface plane $\zeta = 0$. However, Σ^H is taken as the wetted hull surface below the *actual* free surface $\zeta \approx F \, \varphi_\xi$ in the alternative linear flow model—called **Neumann-Michell (NM) flow model**—proposed in [2]. Indeed, the difference between these two wetted hull surfaces yields a *linear* contribution to the hull-surface potential ϕ^H defined by (2) that arguably may not be ignored in a consistent linear flow model. This linear contribution is shown in [1,2] to cancel out the term $G \, \varphi_\xi$ in the line integral around the ship waterline Γ in the NK flow representation (1). Thus, the NK flow representation (1) yields the NM flow representation

$$\phi = \phi^H + F^2 \int_\Gamma d\eta \, (\phi - \varphi) \, G_\xi \;\; . \tag{3}$$

The NK integro-differential equation (1) and the NM integral equation (3) both include a line integral around the ship waterline Γ that involves the flow potential φ.

2. The basic RW (rigid-waterplane) flow representation

In the particular case of a *closed* body, with surface Σ_H, that is *entirely* submerged (at a large or small depth) below the free surface, the classical NK boundary-integral flow representation (1) becomes

$$\phi = \int_{\Sigma_H} da \Big[G \, q^H + (\phi - \varphi) \, \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} G \Big] \;\; . \tag{4}$$

A *special closed submerged body* $\Sigma_H \equiv \Sigma^H_- \cup \Sigma^H_i$ associated with an alternative linear flow model—called **rigid-waterplane (RW) flow model**—is considered in [1,3,4]. In this flow model, a narrow band $-\delta < \zeta \leq 0$ of a free-surface piercing hull surface Σ^H is ignored, and the resulting truncated hull surface Σ^H_- is *closed via a rigid horizontal lid* Σ^H_i where the Neumann boundary condition $\varphi_\zeta = 0$ holds. The portion of the free-surface plane $\zeta = 0$ above the lid Σ^H_i is denoted as Σ^F_i. In the limit $\delta \to 0$, one has $\Sigma^H_- \to \Sigma^H$ and $\Sigma^H_i \to \Sigma^F_i$. The boundary-integral flow representation (4) then becomes

$$\phi = \phi^H + \phi^F_i \;\; \text{where} \;\; \phi^F_i \equiv \int_{\Sigma^F_i} d\xi \, d\eta \, (\phi - \varphi) \, G_\zeta = F^2 \int_{\Sigma^F_i} d\xi \, d\eta \, (\varphi - \phi) \, G_{\xi\xi} \tag{5}$$

and ϕ^H is given by (2). The last expression for ϕ_i^F in (5) follows from the Kelvin-Michell free-surface boundary condition. The RW flow representation (5) involves a surface integral over the ship waterplane Σ_i^F instead of a line integral around the ship waterline Γ and yields an integral equation that determines the flow potential φ over the *extended* hull surface $\Sigma^H \cup \Sigma_i^F$.

3. A 2D flow restriction and the RW-hw and RW-h flow representations

If the flow within the thin water layer $-\delta < \zeta \leq 0$ between the rigid lid Σ_i^H and the waterplane Σ_i^F is assumed to be two-dimensional, the waterplane integral in (5) can be expressed [1,4] as the waterline integral

$$\phi_i^F = -\int_\Gamma d\ell \; A^\Gamma \;\text{ where }\; A^\Gamma \equiv G^\zeta\, q^\Gamma + (\phi - \varphi^\Gamma)\, \boldsymbol{\nu} \cdot \nabla_{\boldsymbol{\xi}} G^\zeta \;\text{ with }\; q^\Gamma \equiv \boldsymbol{\nu} \cdot \nabla_{\boldsymbol{\xi}} \varphi^\Gamma \;. \tag{6}$$

Moreover, $^\zeta$ means integration with respect to ζ, and the unit vector $\boldsymbol{\nu} \equiv (\nu^x, \nu^y, 0)$ is normal to the waterline Γ and points into the water, like the unit vector $\mathbf{n} \equiv (n^x, n^y, n^z)$ normal to Σ^H. The flow representation (5) with ϕ_i^F given by (6) involves both a surface integral over the ship **h**ull surface Σ^H and a line integral around the ship **w**aterline Γ, and accordingly is identified as the **RW-hw flow representation**. The remarkable similarity between the integrands of the waterline-integral representation (6) of ϕ_i^F and the hull-surface integral representation (2) of ϕ^H suggests that numerical cancellations between the components ϕ^H and ϕ_i^F in the RW flow representation (5) can be expected, as is readily verified for a wall-sided ship-hull surface Σ^H. Specifically, the waterline integral (6) can be expressed as the hull-surface integral

$$\phi_i^F = -\int_{\Sigma^H} da\; \partial_\zeta\big(E\, A^\Gamma\big) = -\int_{\Sigma^H} da\, \big(E\, A_\zeta^\Gamma + E_\zeta\, A^\Gamma\big) \;\text{ where }\; E \equiv e^{-9\,\zeta^2/d_*^2} \tag{7}$$

and d_* is a fraction of the nondimensional draft D/L of the ship. The function $E(\zeta)$ vanishes rapidly as $\zeta \to -\infty$ and one has $E(0) = 1$ and $E_\zeta(0) = 0$. Expressions (2) and (7) yield

$$\phi^H + \phi_i^F = \int_{\Sigma^H} da\; A_\Gamma^H \;\text{ where }\; A_\Gamma^H \equiv q^H G + (\phi - \varphi)\, \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} G - E\, A_\zeta^\Gamma - E_\zeta\, A^\Gamma \,. \tag{8}$$

Expressions (8) for A_Γ^H and (6) for A^Γ with the relations $E = 1$ and $E_\zeta = 0$ at Γ yield $A_\Gamma^H = 0$ at Γ. Thus, the integrand A_Γ^H of the surface integral (8) vanishes at the ship waterline Γ, which implies numerical cancellations between the hull-surface integral ϕ^H and the waterplane integral ϕ_i^F in the RW representation (5). The RW flow representation $\phi^H + \phi_i^F$ with ϕ_i^F given by (6) only includes a surface integral over the ship hull surface Σ^H, although the integrand A_Γ^H in (8) involves the flow potential φ^Γ at the waterline Γ, and is then identified as the **RW-h flow representation**.

4. A no-flow restriction and the NN (Neumann-Noblesse) flow representation

The direction of the unit vector $\mathbf{n}$ normal to the body surface $\Sigma_-^H \cup \Sigma_i^H$ defined in the RW flow model is discontinuous along the waterline Γ, and the flow velocity can then be unbounded at Γ. An unbounded flow velocity at Γ can arguably be avoided if the *'no-flow restriction'* $\varphi(\boldsymbol{\xi}) \equiv 0$ is imposed within the thin sheet of water above the rigid lid Σ_i^H. The cancellations between the contributions of the ship-hull surface Σ^H and the ship-waterplane Σ_i^F noted in section 3 arguably also suggest that the restriction $\varphi(\boldsymbol{\xi}) \equiv 0$ if $\boldsymbol{\xi} \in \Sigma_i^F$ may be reasonable. Moreover, the assumption that the thin sheet of water above the rigid lid Σ_i^H is a 'dead-water' region may be argued to imply that the flows around the free-surface piercing open ship-hull surface Σ^H and the related closed body surface $\Sigma_-^H \cup \Sigma_i^H$ are practically identical. Thus, the RW linear flow model with the crucial additional waterplane restriction $\varphi(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \in \Sigma_i^F$ in the limit $\delta = 0$ is considered. As is explained in [1], the restriction $\varphi = 0$ at the ship waterplane Σ_i^F does not necessarily imply that $\varphi = 0$ along the waterline Γ or at the free surface Σ^F outside Γ because the flow potential $\varphi(\xi, \eta, \zeta = 0)$ may be (and likely is) discontinuous across Γ. The basic RW flow representation (5) with the waterplane constraint $\varphi = 0$ becomes

$$\big(1 - C^\Gamma\big)\, \phi = \phi^H \;\text{ where }\; C^\Gamma(\mathbf{x}) \equiv \int_{\Sigma_i^F} d\xi\, d\eta\; G_\zeta = -\int_\Gamma d\ell\; \boldsymbol{\nu} \cdot \nabla_{\boldsymbol{\xi}} G^\zeta \tag{9}$$

and ϕ^H is defined by (2). The waterline-integral representation of the function C^Γ in (9) can be obtained via an elementary mathematical transformation [1]. The boundary integral flow representation (9) is identical to the representation obtained—for wave diffraction-radiation by an offshore structure—in [5] from the usual NK flow model (over 40 years ago) and more recently in [3] from the RW flow model. The flow representation (9) and the RW flow model with the 'no-flow at the ship waterplane' constraint are then identified as the **NN flow representation/model**. The NN flow representation (9) does not include a waterline integral, and indeed does not involve the flow potential φ at Γ. Specifically, the function $C^\Gamma(\mathbf{x})$ defined by the equivalent waterplane or waterline integrals in (9) does *not* involve the flow potential φ. The waterline-integral representation of $C^\Gamma(\mathbf{x})$ in (9) can be evaluated in a straightforward way via the Fourier-Kochin method and expressions (7.56) in [1].

The NN flow representation (9) can be expressed as

$$\phi = \int_{\Sigma^H} da \left[G\, q^H - \varphi\, \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} G \right] + C\, \phi \;\text{ where }\; C \equiv \int_{\Sigma_i^F} d\xi\, d\eta\, G_\zeta + \int_{\Sigma^H} da\, \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} G$$

is the flux through the closed surface $\Sigma_i^F \cup \Sigma^H$ due to a submerged source, or a flux through the free-surface plane, at the singular point $\mathbf{x}$ in the Green function G. One has $C = 0$ if $\mathbf{x}$ is outside the ship-hull surface. Thus, the NN flow representation expresses the potential ϕ at a point $\mathbf{x}$ in the flow region *outside* the ship as

$$\phi \equiv \varphi(\mathbf{x}) = \int_{\Sigma^H} da \left[G\, q^H - \varphi\, \mathbf{n} \cdot \nabla_{\boldsymbol{\xi}} G \right] \tag{10}$$

where the flow potential $\varphi \equiv \varphi(\boldsymbol{\xi})$ at Σ^H is determined by the weakly-singular boundary-integral equation (9). Common displacement ships are streamlined slender bodies for which one has $q^H = F n^x = O(B/L)$ and $\varphi = O(BD/L^2)$ where B and D are the beam or the draft of the ship. Thus, (10) yields

$$\phi \approx F \int_{\Sigma^H} da\; G\, n^x \;. \tag{11}$$

This approximation, proposed by Hogner in 1932 as a composite of Michell's 'thin-ship approximation' and the similar 'flat-ship approximation' proposed by Havelock, is then also an approximate solution of the NN boundary-integral equation (9). The approximation (11) *explicitly* determines the flow created by a ship in terms of the Froude number F and n^x, i.e. the speed and the length of the ship, and the ship-hull form. Indeed, Hogner's approximation is among the most remarkable results in ship hydrodynamics, and is realistic and useful for many practical applications, notably for hull-form optimization, e.g. [6], to analyze the influence of wave interferences on far-field wave patterns, e.g. [7-10], and to filter inconsequential short waves, e.g. [11,12].

5. Conclusions

The waterline integrals in the NK, NM, RW-hw and RW-h flow representations are a difficult issue because the flow potential φ is unlikely to be well defined at the ship waterline Γ. Specifically, although the Neumann boundary condition at Σ^H and the *nonlinear* free-surface boundary condition for steady potential flows are compatible along the actual waterline [13,14], the Kelvin-Michell *linear* free-surface boundary condition at Σ^F and the Neumann boundary condition at Σ^H are *not* compatible at the mean wetted waterline Γ. Thus, the NK, NM, RW-hw and RW-h flow representations can be expected to be ill suited for practical numerical applications; indeed, these flow representations might not be solvable. In particular, numerical difficulties and uncertainties associated with the NK integro-differential equation (1), which has been taken as the theoretical basis of innumerable numerical applications in the past fifty years, are amply reported in the literature. The uncertainties associated with the behavior of the flow potential φ at the ship waterline Γ do not necessarily imply that numerical solutions of the NK, NM and RW flow representations (1), (3) and (5) cannot be obtained via a typical low-order panel method in which the flow potential φ at a waterline segment Γ_m is taken equal to the value of φ at the centroid of the panel Σ_m^H that contains Γ_m. However, attempts to obtain numerical solutions in which φ at Γ is determined as the solution of the boundary-integral flow representation may be expected to result in numerical difficulties. The NN flow representation (9) stands out among the six alternative flow representations given in this study because *it does not involve the flow potential φ at the ship waterline.* This flow representation yields an integral equation that determines φ at the ship hull surface Σ^H.

In the RW flow model, Green's basic identity is applied in the flow region between the submerged closed body surface $\Sigma_-^H \cup \Sigma_i^H$ and the free surface $\Sigma^F \cup \Sigma_i^F$, and the limit $\delta \to 0$ is considered *subsequently.* In short, Green's identity is applied in the flow region that corresponds to $\delta = 0$ in the classical NK problem or to the region that corresponds to $0 < \delta << 1$ in the RW model (in which the limit $\delta = 0$ is considered afterward rather than initially in the NK model). This reversal of the limit $\delta = 0$ yields different boundary-integral flow representations, although the NK and RW flow models satisfy identical boundary conditions if $\delta = 0$. Specifically, the NK, NM, RW, RW-hw, RW-h and NN flow representations satisfy the Laplace equation in the flow region outside the ship-hull surface Σ^H, the Kelvin-Michell linear boundary condition at the undisturbed free surface Σ^F and the Neumann boundary condition at Σ^H, although these alternative flow representations differ significantly. The differences between the NK, NM, RW, RW-hw, RW-h and NN flow representations ultimately stem from the fact that these flow representations correspond to alternative linear flow models. In particular, the ship waterline Γ in the NK flow model is the intersection curve between *two* surfaces: the free surface Σ^F and the mean wetted ship-hull surface Σ^H, where the Kelvin-Michell boundary condition or a Neumann boundary condition are applied. However, in the RW, RW-hw, RW-h and NN flow models, the waterline Γ separates *three* surfaces: the free surface Σ^F, the hull surface Σ^H and the ship waterplane Σ_i^F where three different boundary conditions hold: the free-surface condition, the ship-hull surface condition, and either a '2D-flow' or a 'no-flow' restriction. Differences or incompatibilities between these boundary conditions may result in different local behaviors of the flow potential, and possibly different flow singularities, at Γ.

The 'no-flow' constraint $\varphi = 0$ imposes that the thin sheet of water above the rigid lid that closes a free-surface piercing hull in the RW flow model is a 'dead-water' region, which arguably precludes infinite flow velocities at the waterline and intuitively ensures that the flows around a free-surface piercing ship-hull surface Σ^H and the corresponding submerged body surface $\Sigma_-^H \cup \Sigma_i^H$ defined in the RW flow model are practically equivalent. Lastly, the waterplane constraint $\varphi = 0$ is consistent with the fact that the flow around Σ^H does not determine a flow inside Σ^H, which can then be freely specified and in particular may be chosen nil.

There is no obvious a-priory way of knowing if a flow model, which is merely a theoretical model of real flows, is sufficiently realistic to yield predictions in good agreement with reality; indeed, reasonable approximations, careful analysis, hope and good luck ultimately are major ingredients of any theoretical analysis of fluid flows. Thus, advocacy of a specific boundary-integral flow representation is not the main goal of this brief study. Rather, the primary objective of the study is to consider alternatives to the classical NK linear flow model and the corresponding boundary-integral flow representation (1), which has been steadfastly applied in the past fifty years—despite well-documented numerical difficulties—with limited search for possible alternatives to the NK model and related alternative flow representations. While this study cannot advocate a specific boundary-integral flow representation as was already noted, it can be said that the NN boundary-integral flow representation (9) offers a remarkably simple and appealing alternative to the NK boundary-integral flow representation (1). Moreover, an important feature of the NN flow representation (9) is that it also holds for a ship that steadily advances through regular waves as is shown in [1] and for an offshore structure in regular waves [1,3,5], whereas the classical NK boundary-integral flow representations for an offshore structure in regular waves or a ship that advances in calm water or through waves differ significantly.

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