Lax-Phillips Scattering for Ice Shelf Vibration

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HIGHLIGHTS

Using Lax-Phillips scattering theory, a solution for the hydroelastic vibration of an ice shelf is presented. This serves as a model for the application of this method to other hydroelastic problems.

1 INTRODUCTION

Ice shelves are floating glaciers that form in the Arctic and Antarctic. Recent measurements have shown that waves generated by storms at distant continental coasts impact Antarctic ice shelves [1]. We present here a simple hydroelastic model for ice shelf vibration based on [2, 3]. We show that the problem can be formulated within the paradigm of Lax-Phillips scattering [4] and develop the theory for this. We hope that this serves as a starting point for the application of this theory to hydrolastic problems.

2 MATHEMATICAL MODEL

A shelf of length L and uniform thickness $h \ll L$ floats on a water cavity of uniform depth H_s. The coordinate x denotes horizontal locations along the shelf/cavity, with its origin set to coincide with the seaward end of the shelf and $x = L$ denoting the landward end. Open water of depth H exists for $x < 0$. Figure 1 shows this configuration.

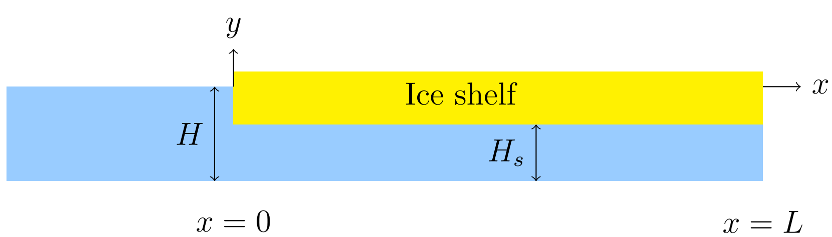

Figure 1: Schematic diagram.

As the wavelengths are assumed to be far greater than the water depth and the wave steepness to be small, the potential satisfies the linear shallow-water equation

$$\partial_x^2 \Phi = \begin{cases} -\frac{1}{H_s}\partial_t \eta, & 0 < x < L, \\ -\frac{1}{H}\partial_t \eta, & x < 0, \end{cases} \tag{1}$$

where $\eta(x,t)$ is the elevation of the water surface, and t denotes time. The function $\Phi(x,t)$ is the velocity potential of the fluid, which satisfies the following no-flux condition at the landward end of the cavity:

$$\partial_x \Phi = 0 \qquad \text{at} \qquad x = L. \tag{2}$$

The ice-shelf is modelled as a thin–elastic plate, meaning its strain field can be determined from the displacement function satisfying

$$-\rho_{\mathrm{w}} \partial_t \Phi = \begin{cases} D\partial_x^4 \eta + \rho_{\mathrm{i}} h \partial_t^2 \eta + \rho_{\mathrm{w}} g \eta & 0 < x < L, \\ \rho_{\mathrm{w}} g \eta & x < 0, \end{cases} \tag{3}$$

where the equation for $x < 0$ is the standard free-surface condition. Here $g \approx 9.81\,\mathrm{m\,s^{-2}}$ is the constant of gravitational acceleration, $\rho_{\mathrm{w}} \approx 1024\,\mathrm{kg\,m^{-3}}$ and ρ_{i} are water and ice densities, respectively, and $D = Eh^3/\{12(1-\nu^2)\}$ is the the flexural rigidity of the shelf, where $E = 11\,\mathrm{GPa}$ is its effective Young's modulus and $\nu \approx 0.33$ its Poisson's ratio. The shelf is clamped at its landward end via the conditions

$$\eta = 0 \quad \text{and} \quad \partial_x \eta = 0 \quad \text{at} \quad x = L, \tag{4a}$$

and free at its seaward end, with conditions

$$\partial_x^2 \eta = 0 \quad \text{and} \quad \partial_x^3 \eta = 0 \quad \text{at} \quad x = 0. \tag{4b}$$

At $x = 0$, the assumption that the draft of the ice shelf is shallow gives rise to the following matching conditions for Φ:

$$\Phi(0^-, t) = \Phi(0^+, t), \quad \text{and} \quad H_s \partial_x \Phi(0^-, t) = H \partial_x \Phi(0^+, t). \tag{5}$$

We rewrite equations (1) and (3) in terms of the negative acceleration potential, $\Psi = -\partial_t \Phi$. We write the non-dimensional equations as an abstract wave , similar to what was done in [5] for a plate on water of finite depth

$$\partial_t^2 \eta + \mathcal{A}^2 \eta = 0, \tag{6}$$

where the operator $\mathcal{A}^2$ is given by

$$\mathcal{A}^2 \eta = \begin{cases} -\dfrac{H_s}{H} \partial_x^2 \Psi, & 0 < x < L, \\ -\partial_x^2 \Psi, & x < 0. \end{cases} \tag{7}$$

The operator $\mathcal{A}^2$ is self-adjoint and positive in the Hilbert space given by

$$\langle \eta, \eta' \rangle_{\mathcal{E}} = \langle \eta, \eta' \rangle_{[-L,\infty]} + \beta \langle \partial_x^2 \eta, \partial_x^2 \eta' \rangle_{[-L,0]}. \tag{8}$$

We want to find the elements of the continuous spectrum of the operator $\mathcal{A}^2$ that are nothing more than single-frequency solutions. We solve

$$\mathcal{A}^2 \overrightarrow{\eta}(x,k) = k^2 \overrightarrow{\eta}(x,k) \tag{9}$$

subject to the conditions that

$$\overrightarrow{\eta}(x,k) = \mathrm{e}^{\imath k x} + R(k)\mathrm{e}^{\imath k x}, \quad x < 0$$

The technique to find $\overrightarrow{\eta}(x,k)$ follows from [3].

3 LAX-PHILLIPS SCATTERING

We introduce a transform given by wave solution $\overrightarrow{\eta}(x,k)$ which is

$$\hat{f}(k) = \frac{1}{2k^2}\left\langle \begin{pmatrix} \eta_0(x) \\ \mathrm{i}\partial_t \eta_0(x) \end{pmatrix}, \begin{pmatrix} \overrightarrow{\eta}(x,k) \\ k\overrightarrow{\eta}(x,k) \end{pmatrix} \right\rangle_{\mathcal{H}} \tag{10}$$

which has inverse

$$\begin{pmatrix} \eta_0(x) \\ \mathrm{i}\partial_t \eta_0(x) \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) \begin{pmatrix} \overrightarrow{\eta}(x,k) \\ k\overrightarrow{\eta}(x,k) \end{pmatrix} \mathrm{d}k \tag{11}$$

This transform allows us to solve the wave equation.

The system has three orthogonal subspaces, $\mathcal{D}_+$ which is the incoming waves, $\mathcal{D}_-$ which is the outgoing waves, and $\mathcal{K}$ which is the solution under the ice-shelf. The space $\mathcal{K}$ is mapped to

$$\mathcal{K} \to H_p^- \ominus R(k) H_p^- \tag{12}$$

where H_p^- is the Hardy space of function analytic in the lower half plane. At this point we use the fact that

$$R(k) = -\prod_{n=-\infty}^{\infty} \frac{k - \bar{k}_n}{k - k_n} \tag{13}$$

noting that this results has not been proven but shown numerically.

Finally we can form a biorthogonal system where

$$\int_{-\infty}^{\infty} \phi_n(k)\psi_m(k)^* \mathrm{d}k = \delta_{mn} \tag{14}$$

with

$$\psi_m = \frac{1}{k - \bar{k}_m} \prod_{n=-\infty}^{\infty} \frac{k - \bar{k}_n}{k - k_n} \frac{1}{\mathrm{d}R/\mathrm{d}k|_{k=k_n}} \tag{15}$$

The solution in the region $\mathcal{K}$ (which is equivalent to the space of functions on $-L < x < 0$) is given by

$$f(k,t) = \sum a_n \psi_n \mathrm{e}^{-\imath k_n t} \tag{16}$$

where

$$\psi_n = \frac{R(k)}{k - \bar{k}_n} \tag{17}$$

We can form an biorthogonal series with

$$\phi_n = \frac{1}{k - k_n} \tag{18}$$

For $f \in \mathcal{K}$ we have

$$f(k) = \sum \frac{R(k)}{k - \bar{k}_n} \frac{f(k_n)}{\mathrm{d}R/\mathrm{d}k|_{k=k_n}} \tag{19}$$

We can see that this implies that in the physical space

$$\eta(x,t) = \sum_n \phi_n(x) \frac{\langle \eta(x,0), \psi_n(x) \rangle_{\mathcal{H}}}{\mathrm{d}R/\mathrm{d}k|_{k=k_n}} \mathrm{e}^{\imath k_n t} \tag{20}$$

where the solutions $\psi_n(x)$ and $\phi_n(x)$ are the analytic continuation of the solutions $\overrightarrow{\eta}(x,k)$ for $k = k_n$ and $k = \bar{k}_n$ respectively.

4 RESULTS AND CONCLUSIONS

We consider typical values for an ice-shelf and compute the analytic extension of the scattering matrix (or reflection coefficient). Figure 2 shows the analytic extension of the scattering matrix (or reflection coefficient) for $L = 40\,\text{km}$, $H = 200\,\text{m}$ and water depth $h = 300\,\text{m}$. Figure 3 shows the time dependent motion starting from a Gaussian displacement of the ice shelf. More results will be shown in the workshop.

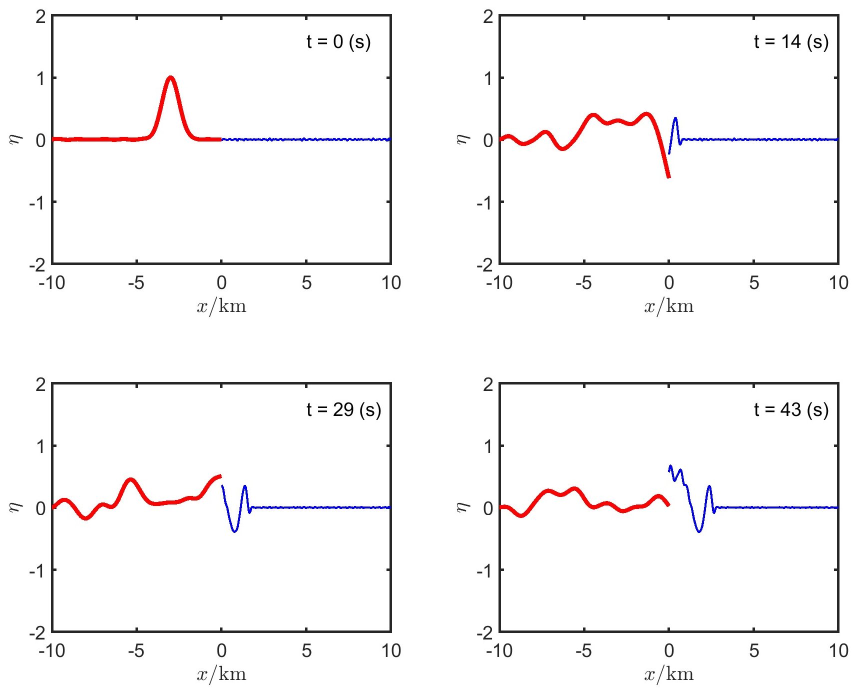

Figure 2: Time-domain results for the vibration of an ice shelf starting from an initial Gaussian displacement. The ice shelf is for $-10 < x < 0$ and open water $x > 0$.

We have shown that a simple model for wave–ice shelf interactions leads to interesting results with important geophysical applications. We hope this work will motivate further study, especially the development of hydroelastic models including more realistic geometries.

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