Practical evaluation of the second order loads in time domain

Malenica S., Blanc A., Jagite G. & De Hauteclocque G.

Bureau Veritas, Paris, France

Introduction

The second order wave loads are induced by the quadratic interactions of the first order quantities (pressures, velocities, wave elevations …). Their evaluation requires a knowledge of the full first order responses. The amplitude of the second order loads in bi-chromatic waves is proportional to the product of the incident wave amplitudes. When formulated in frequency domain, the second order loads occur at frequencies equal to the sum and the difference of the corresponding incident wave frequencies. The generic incident wave field to be considered consists of two waves of different amplitudes (a_i, a_j) , different frequencies (ω_i, ω_j) and different directions (β_i, β_j). It follows that the arbitrary second order quantity $Q^{(2)}(t)$, usually called Quadratic Transfer Function (QTF), can be written as (e.g. see [3]):

$$\begin{aligned} Q^{(2)}(t) = &\quad a_i a_i^* q_0^{(2)}(\omega_i, \beta_i) + a_j a_j^* q_0^{(2)}(\omega_j, \beta_j) \\ &+ \quad \Re\left\{ a_i^2 q_+^{(2)}(\omega_i, \omega_i, \beta_i) e^{2i\omega_i t} + a_j^2 q_+^{(2)}(\omega_j, \omega_j, \beta_j) e^{2i\omega_j t} \right\} \\ &+ \quad 2\Re\left\{ a_i a_j q_+^{(2)}(\omega_i, \omega_j, \beta_i, \beta_j) e^{i(\omega_i+\omega_j)t} + a_i a_j^* q_-^{(2)}(\omega_i, \omega_j, \beta_i, \beta_j) e^{i(\omega_i-\omega_j)t} \right\} \end{aligned} \quad (1)$$

where $q_0^{(2)}$ are called the mean second order quadratic transfer function (QTF), and $q_+^{(2)}$ and $q_-^{(2)}$ are called the sum and the difference frequency quadratic transfer functions. In irregular waves, the time history of the second order forces $F^{(2)}(t)$ is the sum of all possible bi-chromatic wave contributions. One of the problems related to the consistency of the second order loads in frequency domain is related to the fact their evaluation requires the knowledge of the first order motions. This means that the linear body motions cannot be updated during the simulations. This issue represents a nontrivial task, and no fully reliable/practical solution has been produced yet. An interesting proposal was made in [4] where the total second order loading was decomposed into three parts which can be formally written as follows:

$$F^{(2)}(t) = F_{\eta_I\eta_I}^{(2)}(t) + F_{\eta_I\xi}^{(2)}(t) + F_{\xi\xi}^{(2)}(t) \quad (2)$$

where η_I denotes the incident wave elevation and ξ represents the body motions.
The component $F_{\eta_I\eta_I}^{(2)}(t)$ is the function of the quadratic products of the linear "fixed body" quantities (diffraction and incident), the component $F_{\eta_I\xi}^{(2)}(t)$ is the function of the quadratic product of the "fixed body" quantities and the "motion dependent" quantities (radiation), and $F_{\xi\xi}^{(2)}(t)$ is the function of the quadratic products of the linear "motion dependent" quantities. Following similar principles as for the linear case [1], the different time dependent terms are then expressed as a function of the pre-calculated frequency domain quantities and the second order loading is represented in the form of the double convolution integrals. In this way, the linear body motions can be updated during the simulations. The purpose of the present work is to investigate further this approach and make it more practical.

External loading up to second order

The second order theory of wave body interactions are rather well known and has been extensively covered in the literature. We refer to [2] for details, and here we present the final compact expression for the second order loading only:

$$\begin{aligned} \{\boldsymbol{\mathcal{F}}^{(2)}\} = &\iint_{S_{B_0}} \left(P^{(2)}\{\mathbb{N}^{(0)}\} + P^{(1)}\{\mathbb{N}^{(1)}\}\right) dS - \varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(2)}\} + z^{(1)}\{\mathbb{N}^{(1)}\}\right) dS \\ &+ \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma} dC \end{aligned} \quad (3)$$

where the hydrodynamic pressures $P^{(1)}$, $P^{(2)}$ and the wave elevation $\Xi^{(1)}$ are function of the velocity potentials $\Phi^{(1,2)}$.
As it can be seen, both the integral over the mean body surface as well as the integral around the mean body waterline occurs, the second one being due to the integration of the pressure around the mean water level. In the present work, the quadratic products of the first order quantities are of concern only, and the contribution from the pure second order velocity potential $\Phi^{(2)}$ is not considered. We start by noting that the total first order velocity potential $\Phi^{(1)}$ is composed of 8 different components:

$$\Phi^{(1)} = \Phi_I^{(1)} + \Phi_D^{(1)} + \sum_{j=1}^{6} \xi_j \Phi_{Rj}^{(1)} \quad (4)$$

where $\Phi_I^{(1)}$ is the incident potential, $\Phi_D^{(1)}$ is the diffraction potential and $\Phi_{Rj}^{(1)}$ are the six radiation potentials.

This means that the quadratic products of the velocity potentials which are involved in the expression for the second order loading (3) will include three fundamentally different components as suggested by the expression (2). Evaluation of each of those components in the time domain is the main focus of the present investigations.

Hybrid frequency – time domain approach

Instead of solving the problem at each time step, directly in time domain, it is much more convenient to use the hybrid approach which combines the frequency domain results and the integral transforms. In the linear case the method is well established along the principles proposed by Cummins in [1]. The second order approach was not considered very often in the literature and here we refer to the work presented in [4][5].

Linear case

When formulated in frequency domain, it is convenient to use the complex numbers notation so that, for example, the linear velocity potential and the corresponding free surface elevation can be written in the following form:

$$\Phi^{(1)}(\boldsymbol{x},t)=\Re\{\varphi^{(1)}(\boldsymbol{x})e^{i\omega t}\} \quad , \quad \Xi^{(1)}(\boldsymbol{x},t)=-\frac{1}{g}\frac{\partial\Phi^{(1)}}{\partial t}=\Re\{\eta^{(1)}(\boldsymbol{x})e^{i\omega t}\} \tag{5}$$

where $\Phi^{(1)}$ and $\Xi^{(1)}$ are real valued while $\varphi^{(1)}$ and $\eta^{(1)}$ are complex valued.
The Cummins approach which was proposed in [1] is usually applied to the linear radiation forces but it can also be applied to any other linear physical quantity, as for example the wave elevation. In that respect, the wave elevation induced by the body motion in i-th mode $\xi_i^{(1)}$, can be written as:

$$\Xi_i^{(1)}(t)=-\left(\frac{\Re\{\eta_i^{(1)}\}}{\omega^2}\right)^{\infty}\ddot{\xi}_i^{(1)}(t)-\int_0^t K_i^{\eta}(t-\tau)\dot{\xi}_i^{(1)}(\tau)d\tau \quad , \quad K_i^{\eta}(t)=\frac{2}{\pi}\int_0^{\infty}\frac{\Im\{\eta_i^{(1)}\}}{\omega}\cos\omega t\, d\omega \tag{6}$$

where $K_i^{\eta}(t)$ is the memory function associated with the wave elevation induced by the body motion $\xi_i^{(1)}$, $\eta_i^{(1)}$ is the corresponding linear wave elevation induced by the radiation velocity potential $\varphi_{Ri}^{(1)}$ and the superscript " $^{\infty}$" is used to denote that the quantity should be taken at infinite frequency limit.

Second order case

Without loss of generality only the part of the second order loading which is related to the square of the wave elevation is considered. We denote this term by $\{\boldsymbol{\mathcal{F}}_{\xi_i\xi_j}^{t7}(t)\}$ and we introduce the following notation (the superscript " $^{(1)}$" is omitted for simplicity and wall sided body is assumed ($\gamma=0$)):

$$\{\boldsymbol{\mathcal{F}}_{\xi_i\xi_j}^{t7}(t)\}=\frac{1}{2}\varrho g\int_{C_{B_0}}\Xi_i(t)\Xi_j(t)\{\mathbb{N}^{(0)}\}dC \tag{7}$$

After introducing the expressions for the wave elevation Ξ (6), in equation (7) we can rewrite it as follows:

$$\begin{aligned}\{\boldsymbol{\mathcal{F}}_{\xi_i\xi_j}^{t7}(t)\}= \;& \ddot{\xi}_i(t)\ddot{\xi}_j(t)\mathcal{A}_{\xi_i\xi_j}^{t7}+\\ & \ddot{\xi}_i(t)\int_0^t\mathcal{K}_{\xi_i\xi_j}^{t7}(t-\tau)\dot{\xi}_j(\tau)d\tau+\ddot{\xi}_j(t)\int_0^t\mathcal{K}_{\xi_j\xi_i}^{t7}(t-\tau)\dot{\xi}_i(\tau)d\tau+\\ & \int_0^t\int_0^t \hbar_{\xi_i\xi_j}^{t7}(t-\tau_p,t-\tau_q)\dot{\xi}_i(\tau_p)\dot{\xi}_j(\tau_q)d\tau_p d\tau_q\end{aligned} \tag{8}$$

with:

$$\mathcal{A}_{\xi_i\xi_j}^{t7}=\frac{1}{2}\varrho g\int_{C_{B_0}}\left(\frac{\Re\{\eta_i\}}{\omega^2}\right)^{\infty}\left(\frac{\Re\{\eta_j\}}{\omega^2}\right)^{\infty}\{\mathbb{N}^{(0)}\}dC \tag{9}$$

$$\mathcal{K}_{\xi_i\xi_j}^{t7}(t)=\int_0^{\infty}\left[\frac{1}{2}\varrho g\int_{C_{B_0}}\left(\frac{\Re\{\eta_i\}}{\omega^2}\right)^{\infty}\frac{\Im\{\eta_j\}}{\omega}\{\mathbb{N}^{(0)}\}dC\right]\cos\omega t\, d\omega \tag{10}$$

$$\hbar_{\xi_i\xi_j}^{t7}(\tau_p,\tau_q)=\int_0^{\infty}\int_0^{\infty}\mathcal{H}_{\xi_i\xi_j}^{t7}(\omega_p,\omega_q)\cos\omega_p\tau_p\cos\omega_q\tau_q\, d\omega_p d\omega_q \tag{11}$$

$$\{\mathcal{H}_{\xi_i\xi_j}^{t7}(\omega_p,\omega_q)\}=\frac{1}{2}\varrho g\frac{4}{\pi^2}\int_{C_{B_0}}\frac{\Im\{\eta_i(\omega_p)\}}{\omega_p}\frac{\Im\{\eta_j(\omega_q)\}}{\omega_q}\{\mathbb{N}^{(0)}\}dC \tag{12}$$

Here we also recall the equivalent expressions proposed in [4][5] where the method based on 2nd order Volterra approach was used. According to [4][5], in the case of pure radiation, the second order wave elevation can be written as:

$$\left\{\boldsymbol{\mathcal{F}}^{t7}_{\xi_i\xi_j}(t)\right\} = \int_{-\infty}^{t(+\infty)}\int_{-\infty}^{t(+\infty)} h^{t7}_{\xi_i\xi_j}(\tau-\tau_p,\tau-\tau_q)\xi_i(t)(\tau_p)\xi_j(\tau_q)\,d\tau_p d\tau_q \qquad (13)$$

where $h_{\xi\xi}(\tau_p,t_q)$ is the Voltera kernel, which is given by:

$$h^{t7}_{\xi_i\xi_j}(t_p,t_q) = h^{t7+}_{\xi_i\xi_j}(t_p,t_q) + h^{t7-}_{\xi_i\xi_j}(t_p,t_q) \quad , \quad h^{t7\pm}_{\xi_i\xi_j}(t_p,t_q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^{t7\pm}_{\xi_i\xi_j}(\omega_p,\omega_q)e^{i(\omega_p t_p \pm \omega_q t_q)}\,d\omega_p d\omega_q \qquad (14)$$

The functions $H^{t7\pm}_{\xi\xi}(\omega_p,\omega_q)$ are the quadratic transfer functions from frequency domain simulations:

$$H^{t7+}_{\xi_i\xi_j}(\omega_p,\omega_q) = \frac{1}{2}\varrho g\int_{C_{B_0}} \eta_i(\omega_p)\eta_j(\omega_q)\{\mathbb{N}^{(0)}\}dC \quad , \quad H^{-}_{\xi_i\xi_j}(\omega_p,\omega_q) = \frac{1}{2}\varrho g\int_{C_{B_0}} \eta_i(\omega_p)\eta_j^*(\omega_q)\{\mathbb{N}^{(0)}\}dC \qquad (15)$$

where asterix denotes the complex conjugate operation.
Note that the upper integration limit in the integral (13) was not clearly defined in [4] and [5] and that is why we keep both t and $+\infty$. One of the goals of the present work is also to properly define the limits of integration in (13).

Numerical results

The test case is the bottom mounted vertical circular cylinder oscillating in pure surge motion. For that case the semi-analytical solution exists:

$$\varphi_{R1} = \left(f_0(z)B_0H_1(k_0r) + \sum_{n=1}^{\infty} f_n(z)B_nK_1(k_nr)\right)\cos\theta \qquad (16)$$

where $\nu = k_0\tanh k_0H = -k_n\tan k_nH$ and:

$$f_0(z) = \frac{\cosh k_0(z+H)}{\cosh k_0H} \quad , \quad f_n(z) = \frac{\cos k_n(z+H)}{\cos k_nH} \quad , \quad B_0 = \frac{2C_0}{k_0H_1'(k_0a)}\frac{\nu}{k_0^2} \quad , \quad B_n = -\frac{2C_n}{k_nK_1'(k_na)}\frac{\nu}{k_n^2} \qquad (17)$$

and the constants C_i are given by

$$C_0 = \left[2\int_{-H}^{0} f_0^2(z)dz\right]^{-1} = \frac{k_0^2}{H(k_0^2-\nu^2)+\nu} \quad , \quad C_n = \left[2\int_{-H}^{0} f_n^2(z)dz\right]^{-1} = \frac{k_n^2}{H(k_n^2+\nu^2)-\nu} \qquad (18)$$

Linear case

The water depth is 5 meters, cylinder radius is 1 meter, and the irregular motion of the cylinder is considered. The linear results for the 1st order wave elevation induced by the surge motion of the cylinder are shown in Figure 1 where we can observe perfect agreement between the simulated and the reconstructed time histories of the wave elevation.

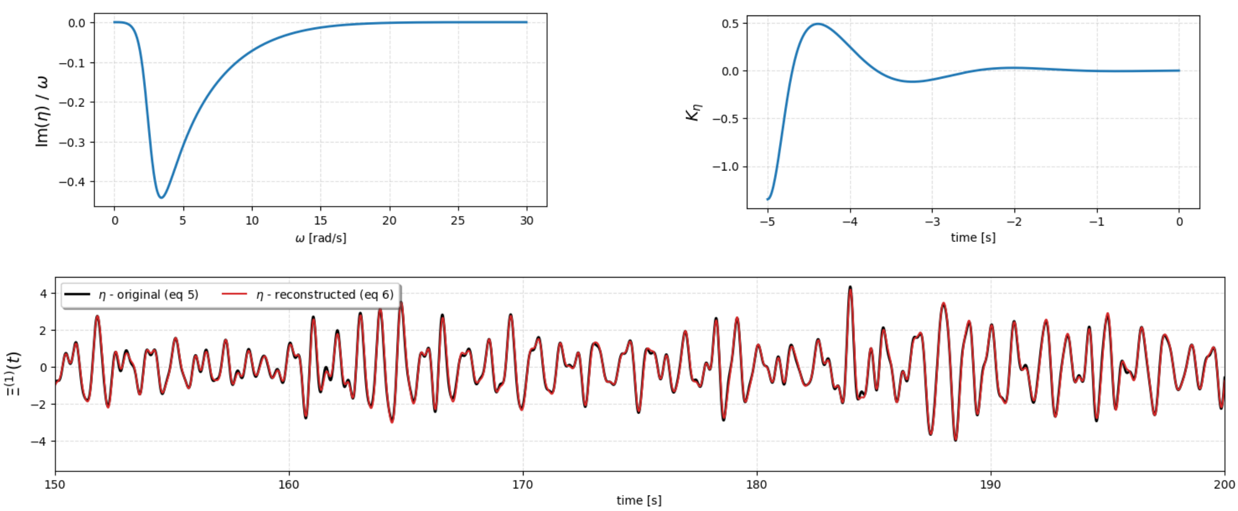

Figure 1: Imaginary component of the local wave elevation (left), corresponding memory function (middle) and the comparisons of the reconstructed and the simulated wave elevation, at upstream point, for irregular cylinder motion.

Second order case

The second order results are summarized in Figure 2 below and again a perfect agreement between the reconstructed and the simulated results is observed. Note that the integral over the waterline (7) was performed from 0 to 60 degrees only, because otherwise, in this particular case, the total integral from 0 to 360 degrees gives zero, due to symmetry of the problem.

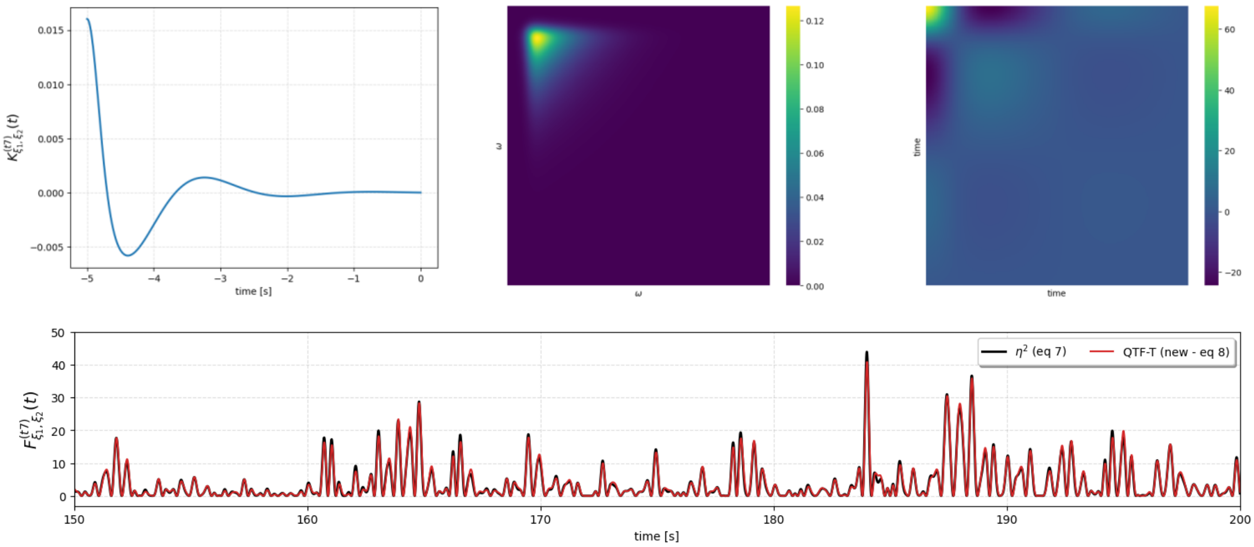

Figure 2: $\mathcal{K}^{t7}_{\xi_i \xi_j}(t)$ (top left), $\mathcal{H}^{t7}_{\xi_i \xi_j}(\omega_p, \omega_q)$ (top middle), $\hbar^{t7}_{\xi_i \xi_j}(\tau_p, \tau_q)$ (top right) and the x component of the second order force $F^{t7}_{x,\xi_i \xi_j}(t)$ (bottom).

Finally, the comparisons of the results obtained by the expressions (8) and (13), are shown in Figure 3, where the important disagreement can be observed. It is important to note that the upper integration limit which has been used in equation (13) is the current time step t and not $+\infty$. When using the upper integration limit of $+\infty$ correct results are obtained, but this is expected because in that case the method reduces to simple application of the double Fourier transformation analysis (direct Volterra approach) which is not of practical interest.

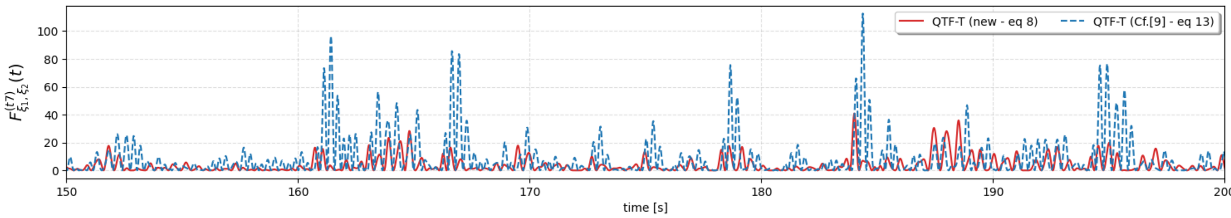

Figure 3: Comparison of the time history of the x component of the second order force $F^{t7}_{x,\xi_i \xi_j}(t)$, obtained using the method proposed here (8) and the expression (13) with the upper integration limit being t.

Discussions and conclusions

Evaluation of the second order loads in time domain using the hybrid frequency-time domain approach has been discussed and the preliminary results have been presented. Both the linear and the second order cases have been validated on the example of the vertical circular cylinder oscillating in surge motion. It has also been shown that the method proposed in [4][5] performs well only if the convolution time covers both the past and future time, which significantly limit the practical application of this method. More results will be presented at the workshop.

References

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