The governing equations for the hydro-acoustic waves in an inviscid compressible fluid with irrotational flows

D. Q. Lu[1, 2, 3, *] and X. Y. Yan[1]

[1] *Shanghai Institute of Applied Mathematics and Mechanics, School of Mechanics and Engineering Science, Shanghai University, Shanghai 200072, China*

[2] *Shanghai Key Laboratory of Mechanics in Energy Engineering, Shanghai 200072, China*

[3] *Shanghai Institute of Aircraft Mechanics and Control, Shanghai 200092, China*

1. Introduction

Hydro-acoustic waves, namely acoustic–gravity waves in the ocean, are low-frequency sound waves at the propagating speed of around 1500 m/s. It can be seen that the research on hydro-acoustic waves is rich and varied. However, the governing equations describing the propagation of hydro-acoustic waves in these studies are various. Most studies directly used the well-known linear wave equation derived by Lighthill (1978, p. 4)[1] as the governing equation for the small-amplitude hydro-acoustic waves, while some studies used varied nonlinear governing equations which stem from the original version derived by Longuet-Higgins (1950)[2]. More recently, some papers by Das & Meylan (2023, 2024)[3, 4], and Pethiyagoda *et al.* (2024, 2025)[5, 6] who added a linear term $g(\partial\Phi/\partial z)$ to the wave equation as their governing equations for the problems they considered, and thought that this term represents the "static compression", where g and Φ are respectively the gravitational acceleration and the velocity potential. However, in our opinion, their viewpoints do not agree with the physical interpretation as we can reveal explicitly the mathematical origin (as shown in § 2). Motivated by the above-mentioned work and issues, we will re-derive the governing equations in details, with a rigorous way, for hydro-acoustic waves in different environmental conditions. The detailed mathematical manipulation demonstrates the physical significance of $g(\partial\Phi/\partial z)$.

2. Mathematical formulation

Establish a three-dimensional Cartesian coordinate system (x, y, z), with the x-axis and the y-axis lying on the horizontal plane, and the z-axis pointing vertically upwards. The equations of continuity and of momentum for a compressible inviscid Newtonian fluid are as follows,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\cdot\boldsymbol{v} = 0, \tag{1}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{f}, \tag{2}$$

where ρ is the fluid density, t the time, $\boldsymbol{v}$ the velocity vector for the fluid flow, p the total pressure (also called thermodynamic pressure), $\boldsymbol{f}$ the body force vector, $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla$. We assume hereinafter the body force is due to the gravity, namely $\boldsymbol{f} = -g\nabla z$.

The convective component in the left-hand side of Eq. (2) can be divided as

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla\frac{|\boldsymbol{v}|^2}{2} - \boldsymbol{v}\times(\nabla\times\boldsymbol{v}). \tag{3}$$

Consider an irrotational flow with $\nabla\times\boldsymbol{v} = \mathbf{0}$. Equation (2) can be simplified as, for an inviscid fluid with an irrotational motion,

$$\frac{\partial\boldsymbol{v}}{\partial t} + \nabla\frac{|\boldsymbol{v}|^2}{2} = -\frac{1}{\rho}\nabla p - g\nabla z. \tag{4}$$

*Corresponding author. Email: dqlu@shu.edu.cn.
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Under the assumption of irrotational flow, $\boldsymbol{v}$ can be expressed in terms of a velocity potential function $\Phi(x,y,z,t)$ as $\boldsymbol{v}=\nabla\Phi$. Substituting it into Eq. (4) yields

$$-\frac{1}{\rho}\nabla p=\nabla\left(\frac{\partial\Phi}{\partial t}+\frac{|\nabla\Phi|^2}{2}+gz\right). \tag{5}$$

According to Longuet-Higgins (1950)[2], the relation connecting the total pressure p and the density ρ is $\dfrac{\mathrm{d}p}{\mathrm{d}\rho}=c^2$, where c is the constant speed of sound in the compressible fluid. Then the left-hand side of Eq. (5) can be rewritten as

$$\begin{aligned}-\frac{1}{\rho}\nabla p&=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x},\frac{\partial p}{\partial y},\frac{\partial p}{\partial z}\right)=-\frac{1}{\rho}\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial\rho}{\partial x},\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial\rho}{\partial y},\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial\rho}{\partial z}\right)\\&=-c^2\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x},\frac{1}{\rho}\frac{\partial\rho}{\partial y},\frac{1}{\rho}\frac{\partial\rho}{\partial z}\right)=-c^2\nabla(\ln\rho)=-\nabla(c^2\ln\rho).\end{aligned} \tag{6}$$

A combination of Eqs. (5) and (6) yields

$$-\nabla(c^2\ln\rho)=\nabla\left(\frac{\partial\Phi}{\partial t}+\frac{|\nabla\Phi|^2}{2}+gz\right), \tag{7}$$

$$\frac{\partial\Phi}{\partial t}+\frac{|\nabla\Phi|^2}{2}+gz+c^2\ln\rho=C(t), \tag{8}$$

where $C(t)$, a function of time, is the Bernoulli constant and can be usually set as zero without loss of generality as demonstrated by Lu (2020)[7]. Therefore,

$$\ln\rho=-\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}+\frac{|\nabla\Phi|^2}{2}+gz\right). \tag{9}$$

In terms of $\boldsymbol{v}=\nabla\Phi$, Eq. (1) can be written as

$$\nabla^2\Phi+\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}=0. \tag{10}$$

In order to derive the governing equation for a compressible fluid, the rate of change of density can be expressed as

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}(\ln\rho). \tag{11}$$

According to Eq. (9) and $\boldsymbol{v}=\nabla\Phi$, the material derivative of $\ln\rho$ can be expanded as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\ln\rho)&=-\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Phi}{\partial t}+\frac{|\nabla\Phi|^2}{2}+gz\right)=-\frac{1}{c^2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)+\boldsymbol{v}\cdot\nabla\left(\frac{\partial\Phi}{\partial t}\right)\right.\\&\qquad\left.+\frac{\partial}{\partial t}\left(\frac{|\nabla\Phi|^2}{2}\right)+\boldsymbol{v}\cdot\nabla\left(\frac{|\nabla\Phi|^2}{2}\right)+\frac{\partial}{\partial t}(gz)+\boldsymbol{v}\cdot\nabla(gz)\right] \qquad (12)\\&=-\frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial|\nabla\Phi|^2}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla|\nabla\Phi|^2+g\frac{\partial\Phi}{\partial z}\right), \qquad (13)\end{aligned}$$

where

$$\boldsymbol{v}\cdot\nabla\left(\frac{\partial\Phi}{\partial t}\right)=\nabla\Phi\cdot\frac{\partial}{\partial t}(\nabla\Phi)=\frac{1}{2}\frac{\partial|\nabla\Phi|^2}{\partial t}, \tag{14}$$

$$\boldsymbol{v}\cdot\nabla(gz)=g\nabla\Phi\cdot\nabla z=g\left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y},\frac{\partial\Phi}{\partial z}\right)(0,0,1)=g\frac{\partial\Phi}{\partial z}. \tag{15}$$

As is shown in Eq. (12), the material derivative of gz can be divided into the local derivative of gz and the convective derivative of gz. Equations (4) and (5) show that gz stems from the body force due to the gravitational acceleration which is always time-independent. z is the time-independent vertical coordinate in the Euler description. Obviously, the local derivative of gz equals zero. Equation (15) shows that the convective derivative of gz yields $g(\partial\Phi/\partial z)$, which represents a coupling between the rate of spatial change of gz and the convective effects of the flow.

Therefore, substitution of Eqs. (11) and (13) into Eq. (10) leads to

$$\nabla^2\Phi - \frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial|\nabla\Phi|^2}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla|\nabla\Phi|^2 + g\frac{\partial\Phi}{\partial z}\right) = 0. \tag{16}$$

Equation (16) is the governing equation for a compressible fluid, satisfying both the conservation laws of the mass and the momentum. It is noted that there is a typographical error in Longuet-Higgins's Eq. (94) (Longuet-Higgins 1950)[2], where $\frac{\partial}{\partial t}(\frac{1}{2}\boldsymbol{v}^2)$ should be replaced with $\frac{\partial}{\partial t}(\boldsymbol{v}^2)$. Equation (16) is equivalent to Longuet-Higgins's corrected Eq. (94). Recently, Kadri & Akylas (2016)[8], Kadri (2019)[9], Michele & Renzi (2020)[10], Kadri & Wang (2021)[11], and Yang & Yang (2024)[12] used Eq. (16) as their governing equations.

Ignoring all nonlinear terms of Eq. (16) yields

$$\nabla^2\Phi - \frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z}\right) = 0. \tag{17}$$

The linear equation (17) was adopted by Abdolali & Kirby (2017)[13], Yang *et al.* (2018)[14], Meza-Valle *et al.* (2023)[15], Das & Meylan (2023, 2024)[3, 4], and Pethiyagoda *et al.* (2024, 2025)[5, 6] as their governing equations. Das & Meylan (2023, 2024)[3, 4], and Pethiyagoda *et al.* (2024, 2025)[5, 6] designed this model (17) to simulate submarine earthquakes and atmospheric pressure gradient, but they thought that $g(\partial\Phi/\partial z)$ represents the "static compression". According to the derivation process above, it is obviously incorrect and will be further discussed at the end of this section.

Next, we assume that the density of the fluid is uniform, namely $\nabla\rho = \mathbf{0}$. Thus there will be no convective derivatives. Then Eq. (10) reduces to

$$\nabla^2\Phi + \frac{1}{\rho}\frac{\partial\rho}{\partial t} = 0. \tag{18}$$

The second term in the left-hand side of Eq. (18) can be expanded as

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}(\ln\rho) = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t} + \frac{|\nabla\Phi|^2}{2} + gz\right) = -\frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{2}\frac{\partial|\nabla\Phi|^2}{\partial t}\right). \tag{19}$$

Here we can see that all convective derivatives in Eq. (12) disappear in Eq. (19). By virtue of Eq. (19), Eq. (18) can finally be written as

$$\nabla^2\Phi - \frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{2}\frac{\partial|\nabla\Phi|^2}{\partial t}\right) = 0. \tag{20}$$

Equation (20) is the governing equation for a compressible fluid with a uniform density, which has not been seen in the literature. Ignoring the nonlinear term in Eq. (20) yields

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0. \tag{21}$$

The linear equation (21), namely the wave equation, is extensively adopted as the governing equation for hydro-acoustic waves.

The difference between two linear equations (17) and (21) is obviously shown with or without $g(\partial\Phi/\partial z)$ which is from the body force term namely the gravity in the equation of momentum (2). Das & Meylan (2023, 2024)[3, 4], and Pethiyagoda *et al.* (2024, 2025)[5, 6] thought that $g(\partial\Phi/\partial z)$ represents the "static compression". But according to the above detailed derivation process, the real difference of Eqs. (17) and (21) is whether or not the density of the fluid is uniform, and whether or not the convective derivative of gz exists. Therefore, in a homogenous fluid, $g(\partial\Phi/\partial z)$ will disappear and cannot at will be added to the wave equation. It is also reasonable to ignore $g(\partial\Phi/\partial z)$ in a stationary fluid or in a linearized problem. Because if the fluid is stationary, namely $\boldsymbol{v}$ is a small quantity, the convective derivative will also disappear.

3. Conclusions

For an inviscid compressible fluid with irrotational flows, we re-derive the governing equations in terms of the velocity potential, as shown in Eq. (16), which is a full mathematical model applicable for the nonlinear motion with the convective effects of incident flow taken into consideration. In view of the detailed mathematical manipulation for Eq. (16), we can find that a highly concerned term $g(\partial\Phi/\partial z)$ is due to the combined effects of the flow convection and the vertical gravitational acceleration, but not the so-called "static compression". This term will disappear for the case of a homogenous fluid, as proved in Eq. (20). Furthermore, in the on-site workshop, the first author will revisit the well-known wave equation obtained for the propagation of small disturbances in a stationary fluid, in a mathematically and physically rigorous way, without the frequently-used assumption that the hydrostatic pressure is constant.

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