Zero reflections of water surface gravity waves caused by a finite periodic array of trapezoidal bars

Huan-Wen Liu

College of Water Conservancy and Civil Engineering, Ludong University, Yantai, PR China

6337@ldu.edu.cn, mengtian29@163.com

HIGHLIGHTS For waves propagating over N periodic widely spaced trapezoidal bars, zero reflections (ZRs) are studied. First, ZRs are classified as **Type-1** and **Type-2**. Second, when the dimensionless bar height, B, with respect to the water depth is very small, both types of ZRs can be explicitly determined, in which the phases of Type-2 ZRs exactly follow the distribution of zero points of a Chebyshev polynomial. Third, when B is not very small, Type-1 ZRs are currently unpredictable, while Type-2 ZRs can be predicted using a simple and accurate formula established here. Clearly, the present study enriches our understanding of Bragg resonance reflection and zero reflection.

1 INTRODUCTION

The phenomenon of ZR of water waves propagating over a bar field means that the field is transparent to incident waves and loses its ability to block water waves. ZR was first studied by Newman [1] who considered wave propagation past a submerged rectangle. He showed that the reflection coefficient, K_R could become zero for suitably chosen values of the rectangle. Mei [2] also studied wave propagation past an underwater rectangle, where both the depths before and after the rectangle may not be equal. He constructed a closed-form solution of K_R to the linear long-wave equation (LLWE), and confirmed those ZRs revealed by Newman [1] once the water depths on both sides of the rectangle are equal. Lin and Liu [3] studied the linear long-wave reflection by a general trapezoid and constructed a closed-form solution of K_R. They found that once a slope is added to a rectangle in front or behind, K_R is no longer zero. Xie et al. [4] studied linear long-wave reflection caused by a rectangle with two scour trenches being attached and a closed-form solution of K_R was derived. It is shown that ZRs occur only when the seabed is symmetric in the wave propagating direction. This revealed that the ZR observed by Newman [1], Mei [2], and Lin and Liu [3] are essentially due to the symmetry of the seabed in the direction of wave propagation.

All the aforementioned ZRs are caused by a single obstacle. Recently, Kar et al. [5] observed a pattern in the number of ZRs on a Bragg bar field, while Xie and Liu [6] revealed the distribution pattern of zero reflections on Bragg bar fields when the bar height is very low. Here we further investigate ZRs caused by a bar field with the bar height being not small. The bar field is composed of N periodically arranged widely spaced trapezoidal bars, see Figure 1, where the water depth profile is

$$h(x) = \begin{cases} h_0 - \frac{2b(x-jd+d)}{w-w_t}, & jd-d \le x \le jd-d+\frac{w-w_t}{2},\ j=1,...,N, \\ h_1, & jd-d+\frac{w-w_t}{2} \le x \le jd-d+\frac{w+w_t}{2},\ j=1,...,N, \\ h_0 + \frac{2b(x-jd+d-w)}{w-w_t}, & jd-d+\frac{w+w_t}{2} \le x \le jd-d+w,\ j=1,...,N, \\ h_0, & \text{otherwise.} \end{cases} \tag{1}$$

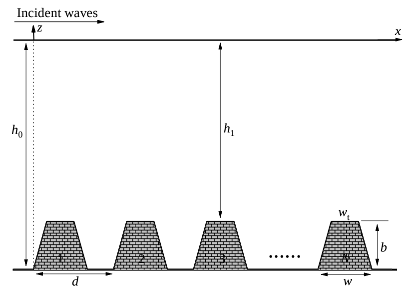

Figure 1: A bar field composed of N widely spaced periodic trapezoidal bars.

2 ZRs CAUSED BY A BAR FIELD WITH THE BAR HEIGHT BEING LOW

If $B = b/h_0 \ll 1$, then based on the regular perturbation, Xie and Liu [6] derived a closed-form solution of K_R as follows

$$K_R = \frac{2BK_0 \left|\sin(W - W_t)\pi \sin(W + W_t)\pi\right|}{\pi(W - W_t)(2K_0 + \sinh 2K_0)} \left|U_{N-1}(\cos 2\pi D)\right|, \tag{2}$$

where $W = \frac{w}{L}$, $W_t = \frac{w_t}{L}$, $D = \frac{d}{L}$, $K_0 = k_0 h_0 = \frac{2\pi}{L} h_0$ with L being the incident wavelength, and $U_{N-1}(\cos 2\pi D)$ is the Chebyshev polynomial of the second kind. Hence all ZRs are

$$\begin{cases} \sin(W - W_t)\pi = 0 \Rightarrow \textbf{Type-1 ZRs:} \quad W - W_t = p, \text{ i.e. } 2D = p\frac{2d}{w-w_t}, \ p = 1, 2, ... \\ \sin(W + W_t)\pi = 0 \Rightarrow \textbf{Type-1 ZRs:} \quad W + W_t = q, \text{ i.e. } 2D = q\frac{2d}{w+w_t}, \ q = 1, 2, ... \\ U_{N-1}(\cos 2\pi D) = 0 \Rightarrow \textbf{Type-2 ZRs:} \quad 2D = n - 1 + \frac{j}{N}, \ j = 1, ..., N-1, \ n = 1, 2, ... \end{cases} \tag{3}$$

Clearly, Type-1 ZR appears in all bar fields with $N \geq 1$ and Type-2 ZR only appears in Bragg bar fields with $N \geq 2$. So they can be regarded as congenital ZR and acquired ZR.

To illustrate them, we take $h_0 = 0.8$ m, $h_1 = 0.76$ m (i.e. $B = 0.05 \ll 1$), $w = 1.6$ m, $w_t = 1.0$ m, $d = 4.03$ m, $N = 1, 2, 3$. K_R is calculated respectively using the closed-form solution [6], i.e., Eq. (2), and the series solution to the modified mild-slope equation (MMSE) [7], see Figure 2. As predicted by Eq. (3), for all N =1, 2, 3, there is only one Type-1 ZR: $2D = \frac{2d}{w+w_t} = 3.1$ in the range $0 < 2D \leq 4$. In addition, for a single bar field, there is no Type-2 ZR, and for the two Bragg bar fields with $N = 2$ and $N = 3$, there are always 1 and 2 Type-2 ZRs between any two adjacent Bragg resonance peaks.

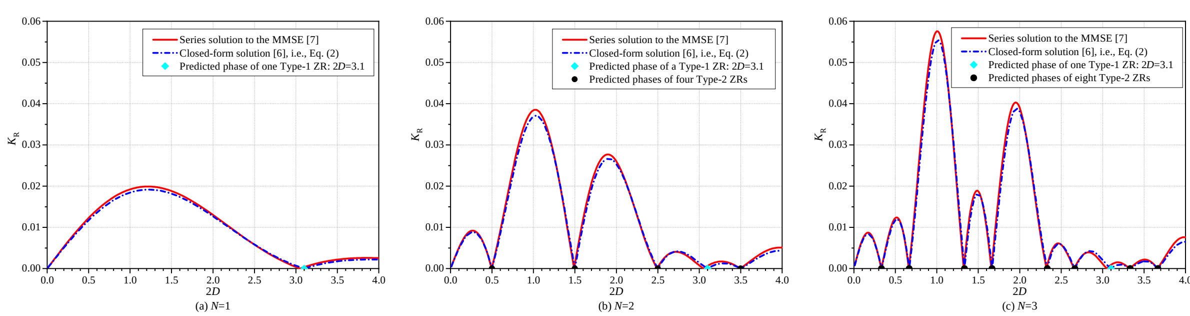

Figure 2: The distribution of ZRs for wave reflection by a trapezoidal bar field with $h_0 = 0.8$ m, $B = 0.05$, $w = 1.6$ m, $w_t = 1.0$ m, $d = 4.03$ m. (a) $N = 1$; (b) $N = 2$; (c) $N = 3$.

3 ZRs CAUSED BY A BAR FIELD WITH THE BAR HEIGHT NOT LOW

When B is not very small, Eq. (2) is no longer valid, and up to date, closed-form solution of K_R has never been found, hence all ZRs cannot be determined directly. Here, we are going to establish a simple formula to predict Type-2 ZRs. Clearly, due to the obstruction of bars, wave propagation will slow down, i.e., $\overline{C} < C_0$, where C_0 is the phase velocity of waves in front of the bar field, and $\overline{C}$ is the average phase velocity between any two adjacent bar centres. According to the modified Bragg's law [6, 8], the n-th Bragg resonance occurs at

$$2D := D_n^{\mathrm{XL}} = \begin{cases} n + 2W - 2W_t\frac{\tanh K_0}{\tanh K_1} - 2(W-W_t)\frac{\epsilon(K_0,K_1)}{K_0 - K_1\frac{\tanh K_1}{\tanh K_0}}, & \text{when } L \text{ is fixed,} \\ \lim_{m\to+\infty} L_{m+1}, & \text{when } d \text{ is fixed,} \end{cases} \tag{4}$$

where $\epsilon(a,b) = a - b - \int_b^a \frac{s\mathrm{e}^{2s}(\mathrm{e}^{2s}+2)}{\mathrm{e}^{4s}-1}\mathrm{d}s$, and L_{m+1} can be calculated using Eqs. (65)-(66) in [8]. Hence, the magnitude of phase downshifting of the n-th order Bragg resonance is

$$\Delta_n = n - D_n^{\mathrm{XL}} = \begin{cases} 2W_t\frac{\tanh K_0}{\tanh K_1} + 2(W-W_t)\frac{\epsilon(K_0,K_1)}{K_0 - K_1\frac{\tanh K_1}{\tanh K_0}} - 2W, & \text{if } L \text{ is fixed,} \\ n - \frac{2d}{\lim_{m\to\infty} L_{m+1}}, & \text{if } d \text{ is fixed,} \end{cases} \tag{5}$$

and Type-2 ZRs between the $(n-1)$-th and n-th Bragg resonances should occur nearly at

$$2D = n - 1 + \frac{j}{N} - \frac{\Delta_{n-1}+\Delta_n}{2},\ j = 1, ..., N-1,\ n = 1, 2, 3, ... \qquad \Delta_0 = 0. \tag{6}$$

First, we consider wave reflection by trapezoidal bars with $h_0 = 0.8$ m, $B = 0.5$, $w = 1.6$ m, $w_t = 0.4$ m, $d = 3.6$ m and N =2, 3. For the two cases, analytical solutions to the MMSE [7] and experimental data [9] are available. According to the modified Bragg's law [6, 8], the predicted phases of the first to fifth order resonances are D_1^{XL}=0.9159, D_2^{XL}=1.8775, D_3^{XL}=2.8879, D_4^{XL}=3.9188, and D_5^{XL}=4.9028, respectively. Correspondingly, $\Delta_1 = 0.0841$, $\Delta_2 = 0.1225$, $\Delta_3 = 0.1121$, $\Delta_4 = 0.0811$, and $\Delta_5 = 0.0972$. According to Eq. (6), in the range $0 < 2D < 5$, for $N = 2$, five Type-2 ZRs should occur at $2D$ =0.4579, 1.3967, 2.3827, 3.4034, 4.4109; and for $N = 3$, ten Type-2 ZRs should occur at $2D$ =0.2912, 0.6246, 1.2300, 1.5634, 2.2160, 2.5494, 3.2367, 3.5701, 4.2441, 4.5775. As we can see from Figure 3 that our prediction of all the Type-2 ZRs coincide with analytical solutions to the MMSE quite good. In addition, it can be seen from Figure 3(a)-(c) that there is one Type-1 ZR at $2D = 4.7461$.

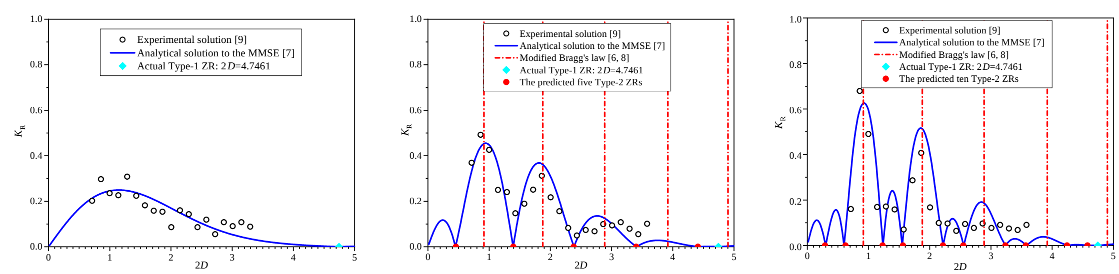

Figure 3: The distribution of Type-2 ZRs for wave reflection by a trapezoidal bar field with $h_0 = 0.8$ m, $B = 0.5$, $w = 1.6$ m, $w_t = 0.4$ m, $d = 3.6$ m. (a) $N = 1$; (b) $N = 2$; (c) $N = 3$.

Next, we consider wave reflection by trapezoidal bars with $h_0 = 2.4$ m, $B = 0.5$, $w = 8.8$ m, $w_t = 4.0$ m, $d = 28.8$ m and N =4, 6. For the two cases, analytical solutions to the MMSE [7] and to the LLWE [10] are available. According to the modified Bragg's law [6, 8], the predicted phases of the first to third order resonances are D_1^{XL}=0.9219, D_2^{XL}=1.8509, and D_3^{XL}=2.7929, respectively. Hence, phase downshifts of the first to third order Bragg resonances are $\Delta_1 = 0.0781$, $\Delta_2 = 0.1491$, and $\Delta_3 = 0.2071$, respectively. According to Eq. (6), in the range $0 < 2D < 3.0$, for $N = 4$, nine Type-2 ZRs should occur at $2D$ =0.2109, 0.4609, 0.7109, 1.1364, 1.3864, 1.6364, 2.0719, 2.3219, 2.5719; and for $N = 6$, fifteen Type-2 ZRs occur at $2D$ =0.1276, 0.2942, 0.4609, 0.6276, 0.7942, 1.0531, 1.2179, 1.3864, 1.5531, 1.7197, 1.9886, 2.1552, 2.3219, 2.4886, 2.6552. As shown in Figure 4, the agreement between our predictions of the Type-2 ZRs and the two analytical solutions [7, 10] is satisfactory. In addition, as shown in Figure 4(a), Type-1 ZR doesn't exist when $0 < 2D < 3.0$.

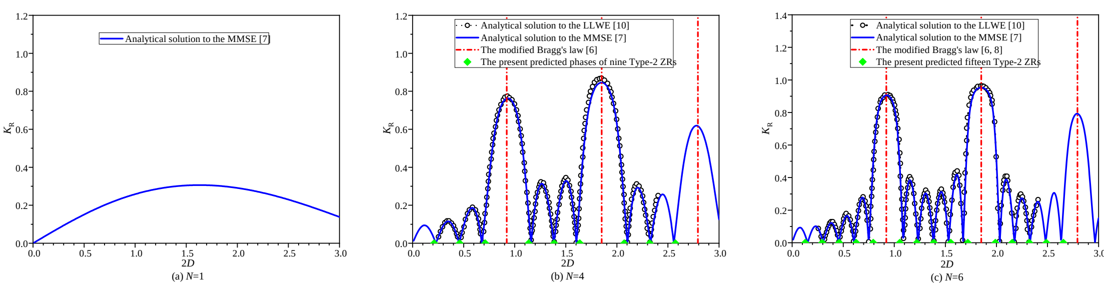

Figure 4: The distribution of ZRs for wave reflection by a trapezoidal bar field with $h_0 = 2.4$ m, $B = 0.5$, $w = 8.8$ m, $w_t = 4.0$ m, $d = 28.8$ m. (a) $N = 1$, (b) $N = 4$; (c) $N = 6$.

REFERENCES

[1] Newman, J. N. 1965. *Propagation of water waves past long dimensional obstacles. Journal of Fluid Mechanics 23*, 23–29.

[2] Mei, C. C. 1989. *The applied dynamics of ocean surface waves.* World Scientific.

[3] Lin, P., and Liu, H.-W. 2005. *Analytical study of linear long-wave reflection by a two-dimensional obstacle of general trapezoidal shape. Journal of Engineering Mechanics-ASCE 131*, 822–830.

[4] Xie, J.-J., Liu, H.-W., and Lin, P. 2011. *Analytical solution for long wave reflection by a rectangular obstacle with two scour trenches. Journal of Engineering Mechanics-ASCE 137*, 919–930.

[5] Kar, P., Sahoo, T., and Meylan, M. 2020. *Bragg scattering of long waves by an array of floating flexible plates in the presence of multiple submerged trenches. Physics of Fluids 32*, 096603.

[6] Xie, J.-J., and Liu, H.-W. 2023. *Analytical study of Bragg resonances by a finite periodic array of congruent trapezoidal bars/trenches on a sloping seabed. Appllied Mathmatical Modelling 119*, 717–735.

[7] Liu, H.-W., Zeng, H.-D., and Huang, H.-D. 2020. *Bragg resonant reflection of surface waves from deep water to shallow water by a finite array of trapezoidal bars. Appled Ocean Research 94*, 101976.

[8] Ding, Y., Liu, H.-W., and Liu, P. 2024. *Quantitative expression of the modified Bragg's law for Bragg resonances of water waves excited by five types of artificial bars. Physics of Fluids 36*, 047130.

[9] Jeon, C.-H., and Cho, Y.-S. 2006. *Bragg reflection of sinusoidal waves due to trapezoidal submerged breakwaters.* Ocean Engineering 33(14-15), 2067–2082.

[10] Chang, H.-K., and Liou, J.-C. 2007. *Long wave reflection from submerged trapezoidal breakwaters.* Ocean Engineering 34(1), 185–191.