Asymptotic Generalised Wagner Model

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1 INTRODUCTION

Two-dimensional symmetric problem of water impact is considered within the so-called Generalised Wagner Model (GWM). This model was introduced by Zhao et al. (1996). The model assumes potential two-dimensional flow. The body shape, the body boundary conditions and the nonlinear Bernoulli equations are original without any simplifications. The dynamic and kinematic boundary conditions on the free surface are linearised and imposed on the horizontal line at the splash-up height. The dynamic boundary condition is reduced to the condition that the velocity potential is equal to zero on the approximated position of the free surface. The splash-up height, which is the vertical coordinate of the intersection point between the free surface and the surface of the body, is calculated using the Wagner condition, Wagner (1932). The Wagner condition states that the elevation of the free surface, which is obtained by the time-integration of the linearised kinematic boundary condition, at the intersection (contact) point is equal to the vertical coordinate of the body surface at this point. The slamming force and the pressure distribution predicted by the GWM agree well with experiments and numerical results by direct simulation of the original nonlinear problem with unknown position of the free surface, see Zhao et al. (1996). The flow velocity in the GWM is singular but integrable at the contact points. Correspondingly, the hydrodynamic pressure at these points is also singular but not integrable. The pressure was suggested to consider only there where it is positive.

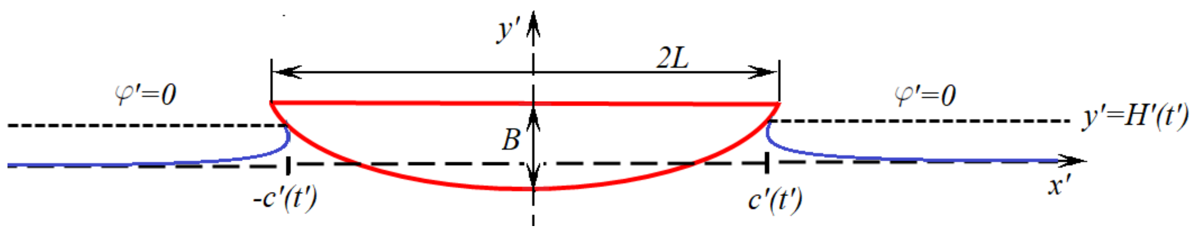

Figure 1: Generalised Wagner model

Zhao et al. (1996) solved the boundary-value problem for the velocity potential and the velocity components at each time step by the boundary element method. The shape of the free surface and the contact point position were obtained by numerical integration of the kinematic boundary condition. Khabakhpasheva et al. (2014) solved the problem using a conformal mapping of the flow region onto the lower half-plane. The conformal mapping was calculated numerically. The Wagner condition was reduced to an integral equation, which was solved numerically. The singularities of the flow and the pressure close to the contact points were separated and incorporated into the algorithm.

In the present study, we consider blunt bodies, for which the GWM problem can be solved approximately using asymptotic methods with a corresponding small parameter. Two-term solution is derived. Its relation to the existing analytical models of water impact, see

Korobkin (2004), is discussed. The same approach can be applied to the original nonlinear problem, but the analysis is more complicated than for the GWM because of more complex conditions on the free surface.

2. FORMULATION OF THE PROBLEM AND ASYMPTOTIC ANALYSIS

The velocity potential $\varphi'(x',y',t')$ of the flow caused by a symmetric blunt body impact onto the initially flat free surface satisfies within the GWM the following equations, see Fig.1,

$$\nabla^2\varphi' = 0 \text{ in } \Omega'(t'), \qquad \varphi'(x', H'(t'), t') = 0 \ \ (y' = H'(t'),\ |x'| > c'(t')),$$
$$\varphi'_{y'} = f'_{x'}(x')\varphi'_{x'} - h'_{t'}(t') \quad (y' = f'(x') - h'(t'),\ |x'| < c'(t')), \tag{1}$$
$$\varphi' \to 0 \ \ (x'^2 + y'^2 \to \infty), \quad \varphi'(x', y', 0) = 0, \quad H'(0) = 0, \quad c'(0) = 0.$$

A prime here stands for dimensional variables, $H'(t') = f'(c') - h'(t')$ is the splash-up hight, the function $f'(x')$ describes the symmetric shape of the impacting body, $f'(0) = 0$, $f'(-x') = f'(x')$, $h'(t')$ is the vertical displacement of the body, $h'(0) = 0$, $h'_{t'}(0) = V_0$ is the initial speed of the body. The shape of the free surface, $y' = \eta'(x', t')$, where $|x'| > c'(t')$, is obtained using the kinematic boundary condition,

$$\eta'(x', t') = \int_0^{t'} \varphi'_{y'}(x', H'(\tau'), \tau')\mathrm{d}\tau'. \tag{2}$$

The Wagner condition reads,

$$\eta'(c'(t'), t') = H'(t'), \tag{3}$$

in the symmetric case. Once the problem (1)-(3) has been solved, the pressure over the wetted part of the body (contact region), $P'(x', t') = p'(x', f'(x') - h'(t'), t')$, is calculated by the non-linear Bernoulli equation, which is written in the form, see Korobkin (2004),

$$P'(x', t') = -\rho\Big(\phi'_{t'} + \frac{f'_{x'}(x')h'_{t'}(t')}{1 + f'^2_{x'}(x')}\phi'_{x'} + \frac{1}{2}\frac{\phi'^2_{x'} - h'^2_{t'}(t')}{1 + f'^2_{x'}(x')}\Big), \tag{4}$$

where $\phi'(x', t') = \varphi'(x', f'(x') - h'(t'), t')$ is the distribution of the velocity potential along the wetted part of the body surface.

For a blunt body, its height B is much smaller than the horizontal dimension L of the body with $\varepsilon = B/L$ being the small parameter of the problem, see figure 1. During the initial stage, when $c'(t') < L$, the dimensionless variables, which are denoted with the same symbols but without the prime, are introduced as

$$x' = Lx, \quad y' = Ly, \quad t' = Bt'/V_0, \quad h' = Bh(t), \quad c' = Lc(t),$$
$$f'(x') = Bf(x), \quad \phi' = LV_0\phi(x, y, t), \quad h' = BH(t), \quad P' = \rho V_0^2 P(x, t). \tag{5}$$

The problem (1) reads in the dimensionless variables (5)

$$\nabla^2\varphi = 0 \text{ in } \Omega(t), \qquad \varphi(x, \varepsilon H(t), t) = 0 \ \ (y = \varepsilon H(t),\ |x| > c(t)),$$
$$\varphi_y = \varepsilon f_x(x)\varphi_x - h_t(t) \quad (y = \varepsilon Y(x, t),\ |x| < c(t)), \tag{6}$$

$$\varphi \to 0 \ (x^2+y^2 \to \infty), \quad \varphi(x,y,0)=0, \quad H(0)=0, \quad c(0)=0.$$

where $Y(x,t)=f(x)-h(t)$ is the current position of the body surface. the Wagner condition (3) takes the form

$$\int_0^t \varphi_y(c(t), \varepsilon H(\tau), \tau)\mathrm{d}\tau = f(c(t)) - h(t). \tag{7}$$

The velocity potential $\varphi(x,y,t,\varepsilon)$ and the function $c(t,\varepsilon)$ depend on the small parameter ε. The potential is sought in the form,

$$\varphi(x,y,t) = \varphi_0(x,y,t) + \varepsilon\varphi_1(x,y,t) + O(\varepsilon^2), \tag{8}$$

as $\varepsilon \to 0$.

Substituting the asymptotic formula (8) in equations (6), (7) and setting $\varepsilon = 0$, we find that the leading order potential $\varphi_0(x,y,t)$ is the velocity potential within the original Wagner model (OWM),

$$\varphi_0(x,0,t) = -h_t(t)\sqrt{c^2-x^2} \quad (|x|<c(t,\varepsilon)),$$
$$\varphi_{0y}(x,0,t) = -h_t(t)(1 - x/\sqrt{x^2-c^2}) \quad (|x|>c(t,\varepsilon)). \tag{9}$$

Note that we do not use the Wagner condition at this stage. The second order potential is convenient to decompose as

$$\varphi_1(x,y,t) = -H(t)\varphi_{0y}(x,y,t) + h_t(t)\hat{\varphi}_1(x,y,t), \tag{10}$$

where

$$\hat{\varphi}_{1y}(x,0,t) = \frac{1}{\pi\sqrt{x^2-c^2}}\int_{-c}^{c}\frac{A(\xi)\mathrm{d}\xi}{\xi - x} \quad (x>c), \tag{11}$$

$$\hat{\varphi}_{1x}(x,0,t) = \frac{1}{\pi\sqrt{c^2-x^2}}\mathrm{P.v.}\int_{-c}^{c}\frac{A(\xi)\mathrm{d}\xi}{\xi - x} \quad (|x|<c), \tag{12}$$

and $A(\xi) = c^2(t,\varepsilon)[f(\xi)-f(c)]/(c^2-\xi^2) + \xi f_\xi(\xi)$. Note that $A(\xi)$ is even function of ξ. For a parabolic shape, $f(x)=x^2$, we have $A(\xi) = 2\xi^2 - c^2$, and for a wedge, $f(x)=|x|$, we have $A(\xi) = \xi - c^2/(\xi+c)$, where $0<\xi<c$. The Wagner condition (7) is approximated with accuracy $O(\varepsilon^2)$ as

$$\int_0^t \Big[\varphi_{0y}(c(t,\varepsilon),0,\tau) + \varepsilon\hat{\varphi}_{1y}(c(t,\varepsilon),0,\tau)\Big]\mathrm{d}\tau = f(c(t,\varepsilon)) - h(t). \tag{13}$$

For a wedge, equations (9), (11) and (13) provide $c(t,\varepsilon) = k(\varepsilon)h(t)$, where $k(\varepsilon) = k_0 + \varepsilon k_1 + O(\varepsilon^2)$, $k_0 = \pi/2$ and

$$k_1 = \frac{1}{\pi}\int_0^\pi G_1(\sin\theta)\mathrm{d}\theta, \ G_1(\sigma) = \log(1-\sigma^2) - \frac{\sigma^2}{1+\sigma}\log\Big(\frac{1-\sigma}{2}\Big) - \frac{\sigma^2}{1-\sigma}\log\Big(\frac{1+\sigma}{2}\Big).$$

Calculations give $k_1 = -0.07944$, which implies that the Wagner approximation $k(\varepsilon) \approx k_0$ is good even for moderate values of ε.

The integral in (11) and (12) reads for a wedge, where $x>0$,

$$\mathrm{P.v.}\int_{-c}^{c}\frac{A(\xi)\mathrm{d}\xi}{\xi - x} = x\log\Big|\frac{c^2-x^2}{x^2}\Big| - \frac{c^2}{c+x}\log\Big|\frac{c-x}{2x}\Big| + \frac{c^2}{c-x}\log\Big|\frac{c+x}{2x}\Big| \tag{14}$$

The integral (14) is denoted as $c\tilde{A}(x/c)$ which is an odd function of x for symmetric shapes.

The dimensionless pressure (5) reads

$$P(x,t) = -\phi_t - \frac{\varepsilon}{2}\frac{\phi_x^2 - h_t^2}{1+\varepsilon^2 f_x^2} - \varepsilon^2 \frac{f_x h_t \phi_x}{1+\varepsilon^2 f_x^2}, \tag{15}$$

where

$$\phi(x,t) = \varphi(x, \varepsilon Y(x,t), t) = -h_t(t)\sqrt{c^2 - x^2} + \varepsilon h_t[f(c) - f(x)] + \varepsilon h_t \hat{\varphi}(x, 0, t) + O(\varepsilon^2). \tag{16}$$

The first two terms in (16) correspond to "GWM in combination with the flat-disc approximation", see Korobkin (2004), eq. (3.25). For a wedge, we obtain $\phi(x,t) = h' c\bar{\phi}(\lambda)$, where $\lambda = x/c(t,\varepsilon)$. Correspondingly the pressure (15) is self-similar, $0 < \lambda < 1$,

$$P(x,t) = -k(\varepsilon)hh''\bar{\phi}(\lambda) + h'^2\Big(k(\varepsilon)\lambda\bar{\phi}_\lambda - k(\varepsilon)\bar{\phi}(\lambda) + \frac{\varepsilon}{2} - \frac{\varepsilon}{2(1+\varepsilon^2)}(\bar{\phi}_\lambda + \varepsilon)^2\Big), \tag{17}$$

$$\bar{\phi}_\lambda = \frac{\lambda}{\sqrt{1-\lambda^2}} + \frac{\varepsilon}{\pi}\frac{\tilde{A}(\lambda)}{\sqrt{1-\lambda^2}} - \varepsilon, \quad \bar{\phi}(\lambda) = -\sqrt{1-\lambda^2} + \varepsilon(1-\lambda) + \frac{\varepsilon}{\pi}\int_{-1}^{\lambda}\frac{\tilde{A}(\xi)\mathrm{d}\xi}{\sqrt{1-\xi^2}},$$

where $\tilde{A}(\lambda)$ is given by (14) for a wedge. The small parameter ε is the tangent of the deadrise angle of a wedge. The pressure distributions for 30^o and 40^o deadrise angles and constant speed of impact are shown in Fig. 2, where the crosses are for fully non-linear numerical solutions by Zhao et al. (1996), the green lines are for the Wagner original model, red lines are for the present model and blue lines are for MLM, which is obtained from (17) by setting $\tilde{A}(\lambda) = 0$. It is seen that the account for higher order terms in the GWM improves the predictions of the impact pressures.

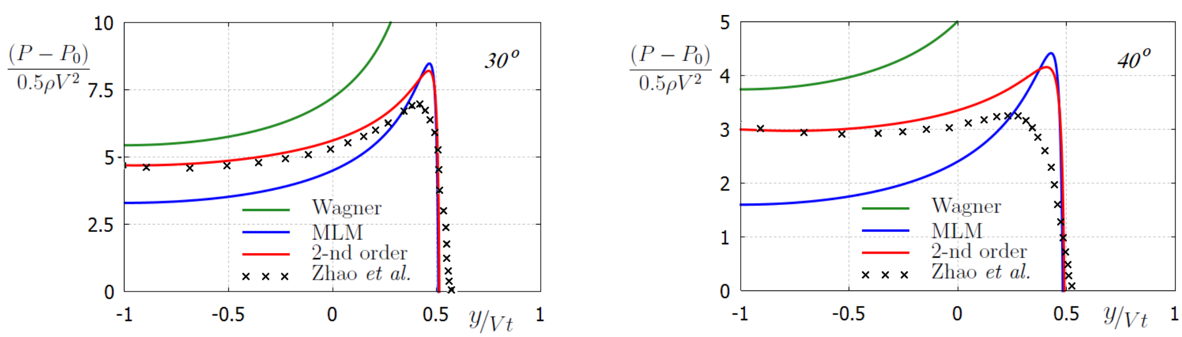

Figure 2: Pressure distribution over the wetted part of a wedge.

Referencies

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