Floating flexible porous thin rectangular plates with free edge conditions exposed to incoming waves in three dimensions

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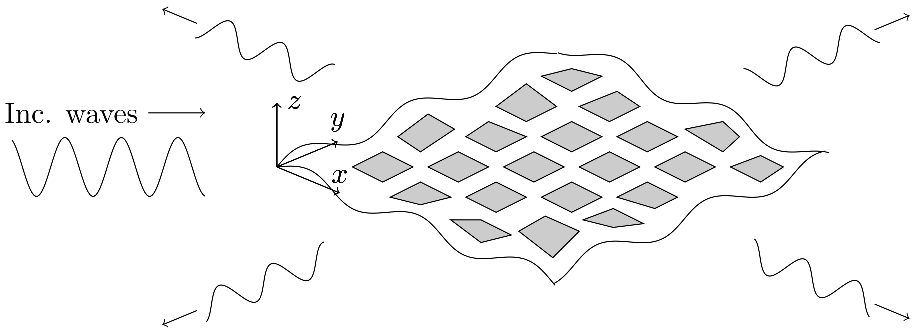


Figure 1: Coordinate system and sketch of a flexible porous thin plate on a wave surface.

Introduction

The theory of flexible plates can be applied to models of floating solar panels, floating airports or for other applications. Vibrational energy gives movement to the wave surface. When wet, the structure vibrates with lower frequencies than when in vacuo. The paper by Korobkin et al. (2023) discusses fundamental issues of free surface and flexible structure interaction in two dimensions. The dry modes and wet modes are properly discussed, mainly in the context of impact. There are no incoming waves.

Our study is concerned with thin rectangular plates with free edge conditions in three dimensions exposed to incoming waves, where investigations seem to be few.

Meylan (1997) derived a variational equation which the plate-liquid system satisfied and presented a solution by the Rayleigh-Ritz method for the forced vibration of a thin plate floating on the half space of an infinite liquid. The dry modes of rectangular plates were recently expressed completely by Liao et al. (2021). We employ these modes to develop a complete hydrodynamical theory of the wave-flexible-structure interaction including the coupled radiation-diffraction problem. The formulation leads to a set of integral equations for the potentials on the wetted side of the flexible plate. The Green function in three dimensions is implemented along the floating geometry using the analysis and practical expansions developed by Newman (1985).

In the present analysis, introducing porosity of the plate represents a damping mechanism of the local vibrations. This is included in the mathematical formulation. The effect is investigated in three dimensions by calculations.

Mathematical formulation

We introduce a Cartesian coordinate system (x, y, z) with the surface on the xy-plane and z vertically. The water depth is infinite. Polar coordinates are applied as $(x, y) = R(\cos\theta, \sin\theta)$ with radial distance $R = \sqrt{x^2 + y^2}$. The motion is assumed linear and time harmonic at the frequency ω of incoming regular waves. The wavenumber is defined by the dispersion relation $k = \omega^2/g$, where g is the gravitational acceleration. Time t and spacial variations are decoupled in the plate-fluid system. Potential flow is applied in the fluid domain, hence the velocity field is expressed as the gradient of the real part of the potential $\Phi(x, y, z, t) = \phi(x, y, z)\mathrm{e}^{i\omega t}$, $i = \sqrt{-1}$.

The modes of the plate. The plate is thin with zero moment and shear forces on the edges. The bending stiffness is

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{1}$$

where E is Young's modulus, h is the plate thickness and ν the Poisson's ratio. The vertical displacement is defined as $W(x, y, t) = w(x, y)\mathrm{e}^{i\omega t}$ and the physical displacement is the real part of W. The Rayleigh-Ritz method is employed, where the displacement is expanded as by Liao et al. (2021)

$$w(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \xi_{ij} f_i(x) g_j(y), \tag{2}$$

with yet unknown modal weights ξ_{ij} and beam functions f_i, g_j along the edges $0 < x < \ell_x$ and $0 < y < \ell_y$, respectively. Rigid beam motions are described by $f_0 = 1$ and $f_1 = \sqrt{3}(2x - \ell_x)/\ell_x$. Hence with $w_{ij} = f_i g_j$, the functions w_{00}, w_{10} and w_{01} gives heave, pitch and roll. The flexible beam functions are

$$f_i(x) = \cos\left(\frac{\beta_i}{\ell_x}x\right) + \cosh\left(\frac{\beta_i}{\ell_x}x\right) - \delta_i\left[\sin\left(\frac{\beta_i}{\ell_x}x\right) + \sinh\left(\frac{\beta_i}{\ell_x}x\right)\right], \qquad \text{for} \quad i = 2, 3, \ldots,$$

with $\delta_i = (\cosh\beta_i - \cos\beta_i)/(\sinh\beta_i - \sin\beta_i)$ and β_i is obtained from the characteristic equation $\cosh\beta_i \cos\beta_i = 1$ for $i = 2, 3, \ldots$. An equivalent formulation exists for $g_j(y)$ with index j. The rigid modes have eigenvalues $\beta_0 = \beta_1 = 0$ for $i, j = 0, 1$. The expansion (2) completely describes the dry modes. The functions $w_{ij} = f_i g_j$ are orthogonal with the inner product $\langle w_{ij}, w_{kl}\rangle = \int_0^{\ell_y}\int_0^{\ell_x} w_{ij} w_{kl} dS = \mathcal{S}$ when $i = k, j = l$ and 0 otherwise. $\mathcal{S} = \ell_x \ell_y$ is the water plane area.

Porous effects. These are modelled by a linear Darcy equation. The relative porous flow is proportional to the pressure jump p across the plate, by Dokken et al. (2017)

$$\alpha p = \frac{\partial \Phi}{\partial z} - \frac{\partial W}{\partial t}, \tag{3}$$

where α is a porosity parameter and $\rho\alpha$ has dimension of time over length, where ρ is the mass density of the fluid.

Radiation and diffraction. The potential is decomposed as $\phi = \phi_D + \phi_R$. Radiation is expressed as $\phi_R = i\omega \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \xi_{ij}\phi_{ij}$. Diffraction is expressed as $\phi_D = \phi_0 + \phi_S$, with $\phi_0 = a_0(ig/\omega)\exp\{-ik(x\cos\theta_0 + y\sin\theta_0) + kz\}$ and ϕ_S describing far-field wave scattering. The following boundary value problem is satisfied: $\nabla^2\phi = 0$ for $-\infty < z < 0$, $-\omega^2\phi + g\partial_z\phi = 0$ at $z = 0$, $\partial_z\phi \to 0$ when $z \to -\infty$, $\phi_{ij} \to R^{-1/2}H_{ij}(\theta)e^{-ikR+kz}$ when $R \to \infty$, for $i, j = 0, 1, \ldots$, $\partial_z\phi_{ij} = w_{ij} - i\omega\rho\alpha\phi_{ij}$ on the plate S_B, $\partial_z\phi_D = -i\omega\rho\alpha\phi_D$ on the plate S_B. The potential is found by using Green's theorem and the following Green function by Newman (1985)

$$G(x, y, z, \bar{x}, \bar{y}, \bar{z}) = \left[r^2 + (z - \bar{z})^2\right]^{-1/2} + kF(X, Z) - 2\pi i k\mathrm{e}^{-Z}\mathrm{J}_0(X), \tag{4}$$

where

$$F(X, Z) = (X^2 + Z^2)^{-1/2} - \pi\mathrm{e}^{-Z}\left(\mathbb{H}_0(X) + \mathrm{Y}_0(X)\right) - 2\int_0^Z \mathrm{e}^{\eta - Z}(X^2 + \eta^2)^{-1/2} d\eta, \tag{5}$$

with $r = (x - \bar{x}, y - \bar{y})$, $X = kr$, $Z = k|z + \bar{z}|$. J_0 and Y_0 are the Bessel functions of zeroth order, of first and second kind, respectively. $\mathbb{H}_0$ is the Struve function of zeroth order, of first kind. We obtain the integral equation for the radiation potentials

$$\int_{S_B} G\left(k\phi_{ij} - w_{ij} + i\omega\rho\alpha\phi_{ij}\right) dS = -2\pi\phi_{ij}(\bar{x}, \bar{y}, \bar{z}), \tag{6}$$

for $(\bar{x}, \bar{y}, \bar{z})$ on the wetted side of the plate S_B. The integral equation accounts for the effects of flexible modes w_{ij} and porosity through the parameter α. The diffraction potential is solved likewise, except for the far field integral which yields an additional term $-4\pi\phi_0$ on the left hand side of the integral equation.

Variational equation and equation of motion. The principle of virtual work applied to the free plate gives the following variational equation by Hildebrand (1965)

$$\delta \int_{S_B} \frac{1}{2} D\left(W_{xx}^2 + W_{yy}^2 + 2\nu W_{xx} W_{yy} + 2(1-\nu) W_{xy}^2\right) dS + \int_{S_B} \rho_p h \frac{\partial^2 W}{\partial t^2} \delta W dS - \int_{S_B} p \delta W dS = 0,$$

with fluid pressure given by linearized Bernoulli equation at the surface $p = -\rho(\partial_t \Phi + gW)$ and ρ_p is the mass density of the plate. Seeking a minimum with respect to the modal weights gives the equation of motion

$$D\xi_{kl}\tilde{K}_{ijkl} - \rho_p h \omega^2 \xi_{kl} \langle w_{ij}, w_{kl} \rangle + i\omega\rho \langle \phi_D, w_{ij} \rangle - \omega^2 \rho \xi_{kl} \langle \phi_{ij}, w_{kl} \rangle + \rho g \xi_{kl} \langle w_{ij}, w_{kl} \rangle = 0, \quad (7)$$

for $i, j, k, l = 0, 1, \ldots$ with definitions

$$\tilde{K}_{ijkl} = E_{ik}^{(2,2)} F_{jl}^{(0,0)} + E_{ik}^{(0,0)} F_{jl}^{(2,2)} + \nu(E_{ik}^{(2,0)} F_{jl}^{(0,2)} + E_{ik}^{(0,2)} F_{jl}^{(2,0)}) + 2(1-\nu) E_{ik}^{(1,1)} F_{jl}^{(1,1)},$$

where

$$E_{ik}^{(r,s)} = \int_o^{\ell_x} \frac{\partial^r f_i(x)}{\partial x^r} \frac{\partial^s f_k(x)}{\partial x^s} dx, \qquad \text{and} \qquad F_{jl}^{(r,s)} = \int_o^{\ell_y} \frac{\partial^r g_j(y)}{\partial y^r} \frac{\partial^s g_l(y)}{\partial y^s} dy, \qquad \text{for } r, s = 0, 1, 2,$$

and we recognize the inner product $\langle w_{ij}, w_{kl} \rangle = E_{ik}^{(0,0)} F_{jl}^{(0,0)}$. Rewriting the equation of motion (7) into

$$\left[K_{ijkl} + C_{ijkl} - \omega^2 M_{ijkl} - \omega^2 A_{ijkl} + i\omega B_{ijkl}\right] \xi_{kl} = \chi_{ij}, \qquad \text{for } \ i, j, k, l = 0, 1, \ldots, \quad (8)$$

where $K_{ijkl} = \sigma \tilde{K}_{ijkl}$ is the stiffness tensor with $\sigma = D/(\rho g)$, $C_{ijkl} = \langle w_{ij}, w_{kl} \rangle$ is the hydrostatic tensor, $M_{ijkl} = \gamma \langle w_{ij}, w_{kl} \rangle$ is the mass tensor with $\gamma = \rho_p h/(\rho g)$. Added mass and damping are defined by $a_{ijkl} = \rho g A_{ijkl}$ and $b_{ijkl} = \rho g B_{ijkl}$ where $-\omega^2 A_{ijkl} + i\omega B_{ijkl} = -\frac{\omega^2}{g} \langle \phi_{ij}, w_{kl} \rangle$, and excitation is defined as $X_{ij} = \rho g \chi_{ij}$, where $\chi_{ij} = -\frac{i\omega}{g} \langle \phi_D, w_{ij} \rangle$.

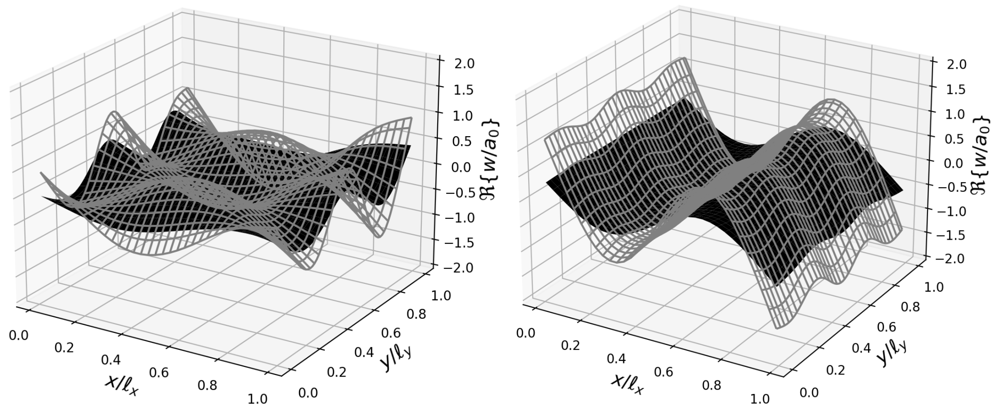
Results

The equation of motion (8) is solved for the weights ξ_{kl}. The response amplitude operators ξ_{ij}/a_0, displacements and vibrations of the plate are obtained. a_0 is the incoming wave amplitude.

The shape of wet modes are independent of forcing and are found by solving the equation of motion as an homogeneous eigenvalue problem. The corresponding eigenvalues of the system represents the wet frequencies. The corresponding eigenvectors of the system represents the wet coefficients used, together with the dry modes of the plate, in the expansion of mode shapes for the wetted plate.

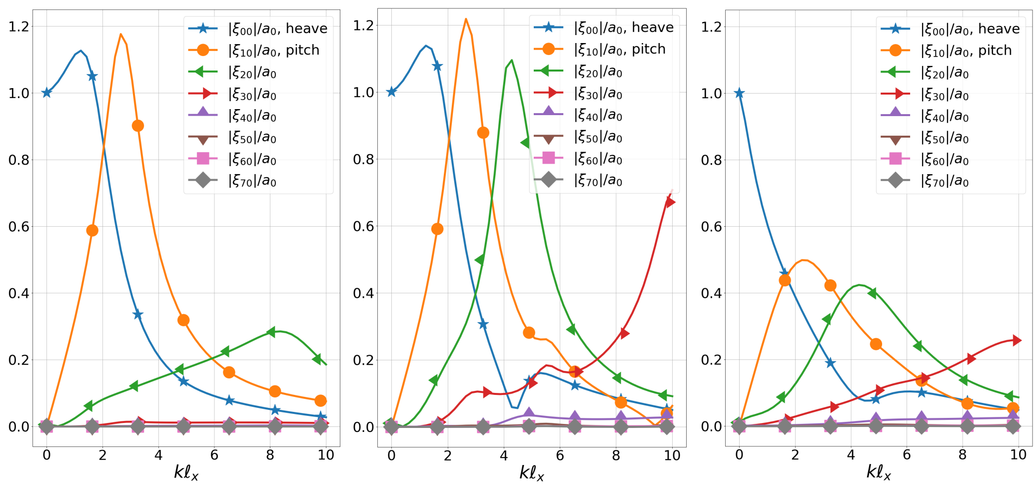
Parameters of interest are, first, the dimensionless bending stiffness $\Pi_1 = (D/\rho g \mathcal{S}^2)$, and, second, the dimensionless porosity $\Pi_2 = \rho\alpha\sqrt{g\ell_x}$. Results are presented for plates with aspect ratios $\ell_x/\ell_y = 1/2$ and $\ell_x/\ell_y = 1$, incoming wave angles of $\theta_0 = 45\,\text{deg}$ and $\theta_0 = 0\,\text{deg}$, soft and stiff plates, with or without porosity.

Figures 2 and 3 show that flexible modes are significant for soft plates. These modes are triggered at relatively shorter wavelengths, which means that not only long waves but also shorter waves exist in flexible structures. Increasing the stiffness of the plate reduces the vertical displacement of the plate, while porosity both dampens and reduces the displacement of the plate. Specifically, increasing the plate stiffness decreases the contribution to the plate's displacement from the flexible modes. On the other hand, increasing the porosity of the plate seems to decrease the contribution of both the rigid and flexible modes, with pitch and roll being reduced significantly. For softer plates, flexible modes are triggered for longer waves.



$\theta_0 = 45\,\mathrm{deg}$ $\theta_0 = 0\,\mathrm{deg}$

Figure 2: $\mathrm{Re}\{w(x,y)/a_0\}$ for $k\ell_x = 10$, $i,j = 0,\ldots,7$, $\ell_x/\ell_y = 1/2$, $\Pi_1 = 6.32\cdot 10^{-5}$, black surface with $\Pi_2 = 0.313$, gray wireframe with $\Pi_2 = 0$ and a_0 is the incoming wave amplitude.



$\Pi_1 = 1.26\cdot 10^{-3}$, $\Pi_2 = 0$ $\Pi_1 = 1.26\cdot 10^{-4}$, $\Pi_2 = 0$ $\Pi_1 = 1.26\cdot 10^{-4}$, $\Pi_2 = 0.313$

Figure 3: $|\xi_{ij}|/a_0$ over $k\ell_x \in (0,10)$, modes along the x-axis is shown with $j=0$ and $i = 0,\ldots,7$, $\ell_x/\ell_y = 1$, $\theta_0 = 0\,\mathrm{deg}$ and a_0 is the incoming wave amplitude.

References

Dokken, J., Grue, J., and Karstensen, L. P. (2017). Wave analysis of Porous Geometry with Linear Resistance Law. J. Marine Sci. Appl., 16: 480-489.

Hildebrand, F. B. (1965). Methods of Applied Mathematics. Prentice-Hall, Inc., Englewood Cliffs, N. J.

Korobkin, A. A., Khabakhpasheva, T. I., and Shishmarev, K. A. (2023). Eigenmodes and added-mass matrics of hydroelastic vibrations of complex structures. J. Fluid Mech., vol. 970, A14.

Liao, C.-Y., Chen, G.-W., Hsu, H.-W., and Ma, C.-C. (2021). Theoretical analysis of vibration characteristics of rectangular thin plate fully immersed in fluid with finite dimensions. Int. J. Mech. Sci., 189, 105979.

Meylan, M. H. (1997). The forced vibration of a thin plate floating on an infinite liquid. J. Sound and vibration, 205(5): 581-591

Newman, J. N. (1985). Algorithms for the free-surface Green function. J. Eng. Math., 19: 57-67.