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Modelling attenuation of waves through broken ice of randomly-varying thickness

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1 Introduction

Attenuation of waves water propagating across regions of randomly-varying water depth is described by localisation (after Anderson (1958)), a multiple scattering effect. Recently, Bennetts *et al.* (2015) considered wave propagation over a randomly-varying bed. They highlighted that theoretical predictions derived from the multiple-scales approach of Mei *et al.* (2005) overestimate the attenuation experienced by individual waves, found by averaging decay of waves in numerical simulations using the step approximation devised by Devillard *et al.* (1988). They determined that the ensemble averaging process used in the analysis of Mei *et al.* (2005) was responsible for phase cancellation of propagating waves.

Very recently, Dafydd & Porter (2024) have considered two-dimensional wave propagation over randomly-varying depth in shallow water in which scattering is governed by a second-order ordinary differential equation (ODE) involving a coefficient of a random function. A modification to the asymptotic multiple-scales approach of Mei *et al.* (2005) removed the phase cancellations in the ensemble averaging not associated with multiple-scattering resulting in a revised theoretical predicion of attenuation that agrees well with direct numerical computations of solutions to the ODE.

Dafydd & Porter (2024) also considered a model for wave propagation through regions of floating broken ice of randomly-varying thickness over water of constant shallow depth and found similar good agreement between theoretical predictions and numerical results. In this work we extend the earlier shallow water modelling and analysis to water of *finite depth*. Within this framework we consider various theoretical and numerical approaches to a model of floating broken ice. The aim of our research is to produce models which could be used to describe the observed attenuation of waves through regions of broken ice in polar regions as described in, for example, Meylan *et al.* (2018).

2 Governing equations

Cartesian coordinates (x, z) are chosen with z directed upwards from the rest position of the water surface in the absence of floating ice. Ice is assumed to entirely cover the surface and is broken into small blocks of width δ which can move only vertically in response to the motion of the underlying fluid but are otherwise unconnected to the motion of neighbouring ice. Fluid occupies $-h < z < -d(x)$ where $d(x)$ is a function that describes submergence of ice. Assuming time-harmonic dependence of angular frequency ω the fluid motion is described by a velocity potential $\mathrm{Re}\left\{\phi(x, z)\mathrm{e}^{-\mathrm{i}\omega t}\right\}$ and the vertical displacement of the ice is given by $\mathrm{Re}\left\{\zeta(x)\mathrm{e}^{-\mathrm{i}\omega t}\right\}$. It follows that ϕ satisfies Laplace's equation throughout the depth so that

$$\nabla^2\phi = 0 \;\text{ on }\; -h < z < -d(x), \tag{2.1}$$

with $\phi_z = 0$ on $z = -h$, the constant bed depth. At the ice/water interface we have the kinematic condition

$$\phi_z = -\mathrm{i}\omega\zeta \quad \text{on} \quad z = -d(x) \tag{2.2}$$

and the dynamic condition which applies over the horizontal interval occupied by each ice block

$$-\omega^2 \rho d(x)\delta\zeta(x) = -\mathrm{i}\omega\rho \int_x^{x+\delta} \phi(x, -d(x))\,\mathrm{d}x + \rho g \delta\zeta(x) \tag{2.3}$$

where δ is the width of each ice block and ρ is the fluid density.

If we assume wavelengths and depths significantly larger than δ we can justify taking the limit $\delta \to 0$ under which (2.3) reduces to $-\omega^2\rho d(x)\zeta = -\mathrm{i}\omega\rho\phi(x, -d(x)) + \rho g\zeta$. Combining this with the (2.2) gives

$$(1 - Kd(x))\phi_z(x, z) = K\phi(x, z), \qquad \text{on} \quad z = -d(x), \tag{2.4}$$

where $K = \omega^2/g$. To determine the effect of varying ice thickness on wave propagation we assume $d(x)$ is randomly varying in $0 < x < L$ and that $d(x) = d_0$ in $x < 0$ and $x > L$, where radiation conditions are expressed in terms of separation solutions as

$$\phi(x, z) = \begin{cases} (\mathrm{e}^{\mathrm{i}k_0 x} + R_L \mathrm{e}^{-\mathrm{i}k_0 x}) Z_0(d_0, z), & x \to -\infty \\ T_L \mathrm{e}^{\mathrm{i}k_0 x} Z_0(d_0, z), & x \to \infty \end{cases}$$

where R_L, T_L are reflection and transmission coefficients, $k_0 = k(d_0)$ is the propagating wavenumber satisfying the dispersion relation for floating ice of constant submergence d_i

$$\frac{K}{1 - Kd_i} = k(d_i) \tanh\left[k(d_i)(h - d_i)\right] \tag{2.5}$$

and vertical eigenfunctions are defined by

$$Z_0(d_i, z) = C_0(d_i) \cosh\left[k(d_i)(h + z)\right] \tag{2.6}$$

where $C_0(d_i)$ is a coefficient which affects only the scaling of the surface displacement.

In $0 < x < L$ we define the ice submergence by the function (Dafydd & Porter (2024))

$$d(x) = d_0(1 + \sigma r(x)) \tag{2.7}$$

where $\sigma \ll 1$ captures the vertical variations of $d(x)$ since $r(x)$ is a random function with zero mean and unit variance, satisfying a Gaussian correlation relation

$$\langle r(x) \rangle = 0, \quad \langle r(x)^2 \rangle = 1, \quad \langle r(x) r(x') \rangle = \mathrm{e}^{-|x - x'|^2/\Lambda^2} \tag{2.8}$$

so that Λ is related to typical horizontal lengthscales of fluctuations in the ice thickness.

We are principally interested in determining the spatial attenuation rate of waves, $\langle k_i \rangle$, averaged over random realisations $r(x)$, and how this depends upon ω, d_0, h and the statistical properties of the ice, Λ and σ.

3 Theoretical prediction for infinitely-long beds

Following the methods of Mei *et al.* (2005), we may use the assumption $\sigma \ll 1$ as the basis of an asymptotic multiple-scales method in which we let $L = \infty$ and write

$$\phi(x, z) \approx \phi_0(x, X, z) + \sigma\phi_1(x, X, z) + \sigma^2\phi_2(x, X, z) + \ldots$$

in $x > 0$ where $X = \sigma^2 x$ is a slow variable. After expanding (2.7) in powers of σ we develop a series of problems at increasing orders of σ which result in

$$\phi(x,z) \approx \left(\mathrm{e}^{\mathrm{i}\left(k_0+\sigma^2\kappa\right)x} + R_\infty \mathrm{e}^{-\mathrm{i}\left(k_0+\sigma^2\kappa\right)x}\right)\mathrm{e}^{-\langle k_i\rangle x} Z_0(d_0,z), \qquad \text{in } x > 0$$

where $|R_\infty| = 1$, $\kappa \in \mathbb{R}$ is a known but unimportant phase factor and

$$\langle k_i \rangle = \frac{\frac{\sqrt{\pi}}{2} {d_0}^2 {k_0}^4 \sigma^2 \Lambda \mathrm{e}^{-{k_0}^2\Lambda^2}}{\left(k_0(h-d_0)\mathrm{sech}^2\left[k_0(h-d_0)\right] + \tanh\left[k_0(h-d_0)\right]\right)^2} \tag{3.1}$$

plus small corrections from evanescent contributions. The formula (3.1) agrees with Dafydd & Porter (2024) in the shallow-water limit where the peak in the attenuation lies at $k_0\Lambda \approx 1$, while in the deep-water limit we see the peak at $k_0\Lambda \approx \sqrt{2}$.

The expression (3.1) is derived from a solvability condition at second order in σ which involves taking an ensemble average over the random function $r(x)$ and requires some care to avoid including incoherent phase cancellations.

4 Step approximations and a limiting ordinary differential equation

In order to test (3.1) we simulate solutions over long ($L \gg d_0$) section of varying-thickness ice by calculating solutions numerically for a large number $N = O(500)$ random functions $r(x)$ and ensemble averaging the numerically-determined of k_i after each scattering calculating to produce $\langle k_i\rangle$. We have developed two (related) numerical methods. The first is a step approximation (as in Devillard *et al.* (1998)) in which $d(x)$ is discretised into a piecewise continuous function of width δ and scattering is determined from the product of transfer matrices that describe the local scattering at the discontinuities in $d(x)$. Each transfer matrix encodes the relationship between wave amplitudes to the left (A_1, B_1) and right (A_2, B_2) of single jump in ice thickness from d_1 to d_2 at the origin (say) in isolation of all others, so that

$$\phi(x,z) = \begin{cases} \left(A_1 \mathrm{e}^{\mathrm{i}k_1 x} + B_1 \mathrm{e}^{-\mathrm{i}k_1 x}\right) Z_0(d_1,z), & x \to -\infty \\ \left(A_2 \mathrm{e}^{\mathrm{i}k_2 x} + B_2 \mathrm{e}^{-\mathrm{i}k_2 x}\right) Z_0(d_2,z), & x \to \infty \end{cases} \tag{4.1}$$

where $k_i = k(d_i)$ for $i = 1, 2$ are the roots of (2.7). The solution to this problem is possible using separation solutions in $x < 0$ and $x > 0$ and matching across $x = 0$. For sufficiently small jumps, equivalent to suitably small discretisations, the integral equations that result can be approximated accurately using a one-term variational approximation, as described in Porter (2020).

The step-approximation formulation is also used as the basis of the method outlined in Porter (2020) in which we can take the limit as the step size, δ, goes to zero. The details are complicated, but the process results in a continuum model in which discrete changes in d_i and k_i are replaced by continuous functions and the particular choice ${C_0}^2(d) = 2/(k(2k(h-d) + \sinh[2k(h-d)]))$ results in a simplified depth-averaged model (a mild-slope equation) for wave propagating through broken ice governed by the second order ODE

$$(k^{-2}\Gamma')' + \Gamma(x) = 0, \tag{4.2}$$

where $k = k(d(x))$ is defined locally by (2.7) and $\Gamma(x)$ is a function related to $\zeta(x)$. Thus, we have also computed numerical solutions to (4.2) for random $r(x)$ to determine ensemble averages, $\langle k_i\rangle$, of the attenuation from this model.

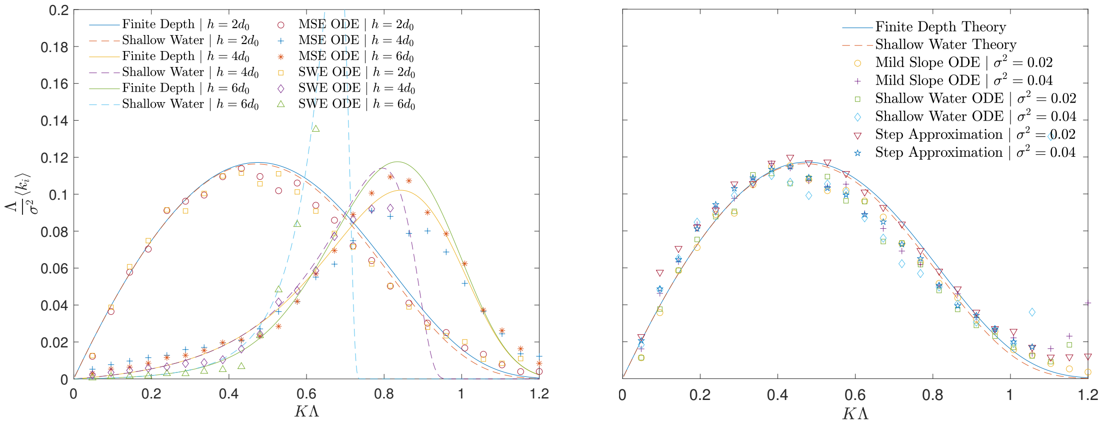

Figure 1: Scaled ensemble averaged attenuation coefficients for $N = 500$ simulations for length $L = 10\Lambda/\sigma^2$, average submergence $d_0 = 1$ and correlation length $\Lambda = 2d_0$. Left: $\sigma^2 = 0.02$, Right: $h = 2d_0$

5 Comparison of models

In Figure 1 curves showing the theoretical prediction (3.1) are compared with computations from the step approximation and the limiting mild-slope equation for varying ice on finite water depth. Overlaid are results taken from Dafydd & Porter (2024) for shallow water. In the right-hand panel the fluid depth is shallow ($h = 2d_0$) and simulations are made with different values of σ^2 (strength of randomness). In the left-hand panel we vary the depth of the fluid for fixed σ^2 to show how the finite-depth results diverge from the shallow water scaling $\Lambda\langle k_i\rangle/\sigma^2 \sim {k_0}^2\Lambda^2 \mathrm{e}^{-{k_0}^2\Lambda^2}[d_0/(h-d_0)]^2$ towards a deep water scaling $\langle k_i\rangle\Lambda/\sigma^2 \sim {k_0}^4 d_0^2\Lambda^2\mathrm{e}^{-{k_0}^2\Lambda^2}$. Further results will be shown at the Workshop including comparisons to published field data.

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