Wave scattering by an annular metamaterial cylinder consisting of curved plates

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Abstract

We investigated wave interaction with an annular metamaterial cylinder, which is composed of a series of curved plates extending through the depth. Fluid can flow tangentially between the curved plates. Two physical findings are reported here. First, the free surface pattern outside the metamaterial cylinder is remarkably symmetric about the axis of incidence of plane waves. Second, the field enclosed by the annular cylinder is also symmetrical, nevertheless, the plane of symmetry is deflected from the incident plane waves, and the deflection angle is determined by the angle difference between the two ends of any specified plate.

Mathematical model

In this model, an annular metamaterial cylinder subjected to unidirectional regular waves in water of finite depth h is considered (see Fig. 1). The annular cylinder with its inner and outer radii denoted as R_i and R, respectively, is composed of a periodic array of infinitely–thin vertical curved plates. A global Cartesian coordinate system $Oxyz$ is chosen with the mean free–surface coinciding with the (x, y)–plane, Oz placed at the axis of the cylinder pointing vertically upwards, and Ox coinciding with the incident wave direction. Hence the fluid bottom is at $z = -h$. Moreover, a cylindrical coordinate system, $Or\theta z$, is chosen for the purpose of convenience of mathematical expression. Fluid is allowed to flow in gaps between adjacent plates and waves are supported by the free surface. The effect of these plates of the annular cylinder allows waves to propagate along the curved channels between the plates.

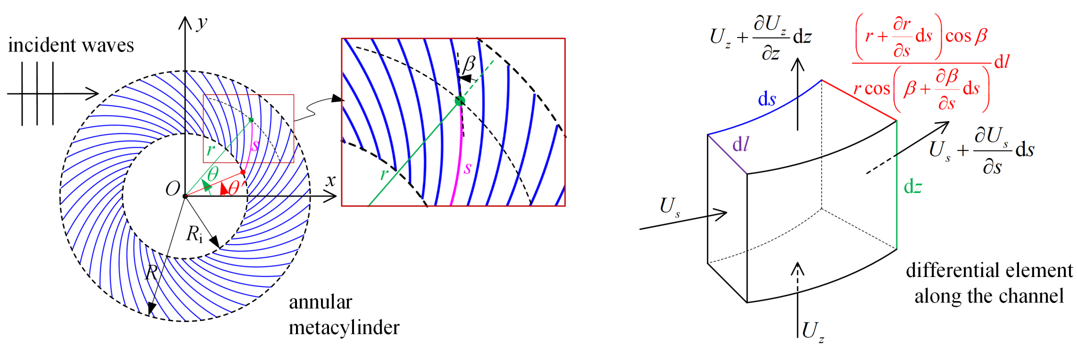

Figure 1: Schematic of an annular metamaterial cylinder consisting of curved plates (left). s denotes the length of the plate starting from $(r, \theta) = (R_\mathrm{i}, \theta')$ to $(r, \theta) = (r, \theta' + \mu(r))$, where $\mu(r)$ represents the rotation angle of the plate at r. $\beta(r)$ denotes the angle of the tangent on the plate relative to the radial direction. A differential element along the channel (right).

We assume that all amplitudes are small enough that linear theory applies and we make the usual assumptions that the fluid is inviscid, incompressible and its motion is irrotational. We denote the fluid velocity potential by $\Phi(x, y, z, t)$. It is further assumed that all motion is time–harmonic with angular frequency ω. Thus, we can write $\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, y, z)\mathrm{e}^{-\mathrm{i}\omega t}\}$, where Re denotes the real part and ϕ is the spatial velocity potential which is independent of time, i.e., t. i is the imaginary unit.

The fluid domain can be divided into an annular domain, which fills the annular metamaterial cylinder, and an exterior domain, representing the fluid domain outside the metamaterial cylinder extending towards infinity horizontally, and an inner region. The complex velocity potential ϕ is subjected to the following boundary value problem [1]

$$\nabla^2\phi = 0 \qquad \text{in the water} \tag{1}$$

$$\phi_z - \omega^2\phi/g = 0 \qquad \text{on } z = 0 \tag{2}$$

$$\phi_z = 0 \qquad \text{on } z = -h \tag{3}$$

$$\sqrt{r}\left(\frac{\partial}{\partial r} - \mathrm{i}k\right)(\phi - \phi_I) = 0 \qquad \text{when } r \to \infty \tag{4}$$

where k is the wavenumber in finite water depth satisfying the gravity wave dispersion relation $\omega^2 = gk\tanh(kh)$, g denotes the acceleration due to gravity, and ϕ_I represents the velocity potential of incident waves. Within the fluid in the annular cylinder, Eq. (1) also holds although it is confined to narrow disconnected domains bounded by thin plates aligned with the radial direction of the $Or\theta z$ coordinate.

Figure 1b shows a differential element along the channel. The fluid is assumed to be imcompressable, hence we have

$$\begin{aligned}&\frac{\partial U_z}{\partial z}\mathrm{d}z\frac{\mathrm{d}s}{2}\left(\mathrm{d}l + \frac{\left(r + \frac{\partial r}{\partial s}\mathrm{d}s\right)\cos\beta}{r\cos\left(\beta + \frac{\partial \beta}{\partial s}\mathrm{d}s\right)}\mathrm{d}l\right)\\ &\qquad + \left(U_s + \frac{\partial U_s}{\partial s}\mathrm{d}s\right)\mathrm{d}z\frac{\left(r + \frac{\partial r}{\partial s}\mathrm{d}s\right)\cos\beta}{r\cos\left(\beta + \frac{\partial \beta}{\partial s}\mathrm{d}s\right)}\mathrm{d}l - U_s\mathrm{d}l\mathrm{d}z = 0,\end{aligned} \tag{5}$$

which further gives

$$\frac{\partial U_s}{\partial s} + \frac{\partial r}{\partial s}\frac{U_s}{r} + \frac{\partial U_z}{\partial z} = 0, \tag{6}$$

where U_s and U_z denote the horizontal and vertical fluid velocities, respectively, and

$$r(s) = R_\mathrm{i} + \int_0^s \cos\beta'(s')\mathrm{d}s', \qquad \frac{\partial r}{\partial s} = \cos\beta'(s), \tag{7}$$

where $\beta'(s) = \beta(r)$. We can obtain the reduced Laplace's equation by using $U_s = \frac{\partial\phi}{\partial s}$, $U_z = \frac{\partial\phi}{\partial z}$, and Eq. (7) to Eq. (6)

$$\frac{\partial^2\phi}{\partial s^2} + \frac{\cos\beta'(s)}{R_\mathrm{i} + \int_0^s \cos\beta'(s')\mathrm{d}s'}\frac{\partial\phi}{\partial s} + k^2\phi = 0, \tag{8}$$

which governs the fluid motion at the annular region occupied by the metamaterials.

The spatial velocity potential in the exterior domain, $r \in [R, \infty)$, can be expressed as

$$\phi_{ext} = \phi_I + Z_0(z)\sum_{m=-\infty}^{\infty} A_m H_m(kr)\mathrm{e}^{\mathrm{i}m\theta}, \tag{9}$$

where the second term denotes the components contributed by the waves scattered from the cylinder. A_m are the unknown coefficients to be determined; H_m denote the Hankel functions of the first kind of order m; $Z_0(z) = \cosh[k(z+h)]/\cosh(kh)$

The velocity potential at the interior domain, $r \in [0, R_\mathrm{i}]$, can be written as

$$\phi_{int}(r,\theta,z) = Z_0(z) \sum_{m=-\infty}^{\infty} B_m J_m(kr)\mathrm{e}^{\mathrm{i}m\theta}, \tag{10}$$

where B_m are the unknown coefficients to be determined.

General solutions of the reduced Laplace's equation, i.e., Eq. (8), at the annular domain ($r \in [R_\mathrm{i}, R]$) satisfying free surface and bed boundary conditions may be expressed as

$$\phi_{ann}(s,\theta',z) = Z_0(z)\left[C'(\theta')F'^{(1)}(s) + D'(\theta')F'^{(2)}(s)\right], \tag{11}$$

in which $F'^{(1)}(s)$ and $F'^{(2)}(s)$ are two (linearly independent) solutions of the reduced Laplace's equation, and they are dependent on the shape of the curved plates.

After mapping θ' and s into (r,θ) and r, respectively, Eq. (11) may be rewritten as

$$\phi_{ann}(r,\theta,z) = Z_0(z)\left[C(r,\theta)F^{(1)}(r) + D(r,\theta)F^{(2)}(r)\right], \tag{12}$$

in which $F^{(1)}(r)$ and $F^{(2)}(r)$ are two independent solutions of the reduced Laplace's equation

$$\cos^2\beta\frac{\partial^2\phi}{\partial r^2} + \cos^2\beta\frac{\partial\phi}{r\partial r} + k^2\phi = 0, \tag{13}$$

which is obtained after inserting Eq. (7) into Eq. (8).

The functions $C(r,\theta)$ and $D(r,\theta)$ can be expanded as

$$C(r,\theta) = \sum_{m=-\infty}^{\infty} C_m \mathrm{e}^{\mathrm{i}m[\theta-\mu(r)]},\ \ D(r,\theta) = \sum_{m=-\infty}^{\infty} D_m \mathrm{e}^{\mathrm{i}m[\theta-\mu(r)]}, \tag{14}$$

where C_m and D_m are the unknown coefficients to be determined; $\mu(r)$ denotes the angular position of a point (i.e., (r,μ)) on the channel, which merges at $(R_\mathrm{i}, 0)$ on the inner edge of the annular region (see Fig. 1), and it can be expressed as

$$\mu(r) = \int_{R_\mathrm{i}}^{r} \frac{\tan\beta}{r}\mathrm{d}r. \tag{15}$$

The final expression of the velocity potential at the annular region can be written as

$$\phi_{ann} = Z_0(z)\sum_{m=-\infty}^{\infty}\left[C_m F^{(1)}(r) + D_m F^{(2)}(r)\right]\mathrm{e}^{\mathrm{i}m[\theta-\mu(r)]}. \tag{16}$$

The unknown coefficients A_m, B_m, C_m, and D_m can be determined by matching the continuity of the pressure and flux across $r = R_\mathrm{i}$ and R. The symmetry of the water surface in the exterior and interior domains is theoretically proved. For the sake of limited space, detailed equations are not presented here.

Results

We take the following three cases (see Table 1 and Fig. 2) as an example to demonstrate the interaction of waves with an annular metamaterial cylinder consisting of curved plates. In different cases, the shape of the curved plates is governed by different types of functions.

Table 1: Definition of the parameters/functions associated with the three cases

Case No.	$\cos\beta$	$\mu(r)$	$F^{(1)}(r)$	$F^{(2)}(r)$
1	a_1	$\tan\beta \ln \frac{r}{R_\mathrm{i}}$	$J_0\left(\frac{kr}{a_1}\right)$	$Y_0\left(\frac{kr}{a_1}\right)$
2	a_2/r	$\tan\beta(r) - \tan\beta(R_\mathrm{i}) - \beta(r) + \beta(R_\mathrm{i})$	$J_0\left(\frac{kr^2}{2a_2}\right)$	$Y_0\left(\frac{kr^2}{2a_2}\right)$
3	$a_3\sqrt{r}$	$2(-\tan\beta(r) + \tan\beta(R_\mathrm{i}) + \beta(r) - \beta(R_\mathrm{i}))$	$J_0\left(\frac{2k\sqrt{r}}{a_3}\right)$	$Y_0\left(\frac{2k\sqrt{r}}{a_3}\right)$

Figure 3 illustrates the instantaneous wave motion for these three cases. It is observed that the free surface pattern outside the metamaterial cylinder is remarkably symmetric about the axis of incidence of plane waves. Moreover, the field enclosed by the annular cylinder is also symmetrical, nevertheless, the plane of symmetry is deflected from the incident plane waves, and the deflection angle is $\pi/4$, i.e., the angle difference between the two ends of any specified plate.

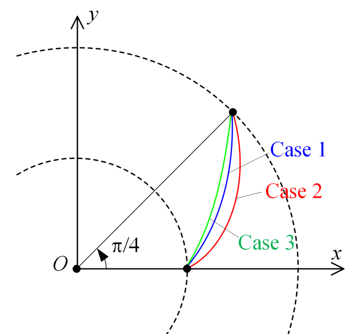

Figure 2: Shapes of the curved plate for the three examined cases with $R/h = 1.0$, $R_\mathrm{i}/R = 0.5$, and $\mu(R) = \pi/4$.

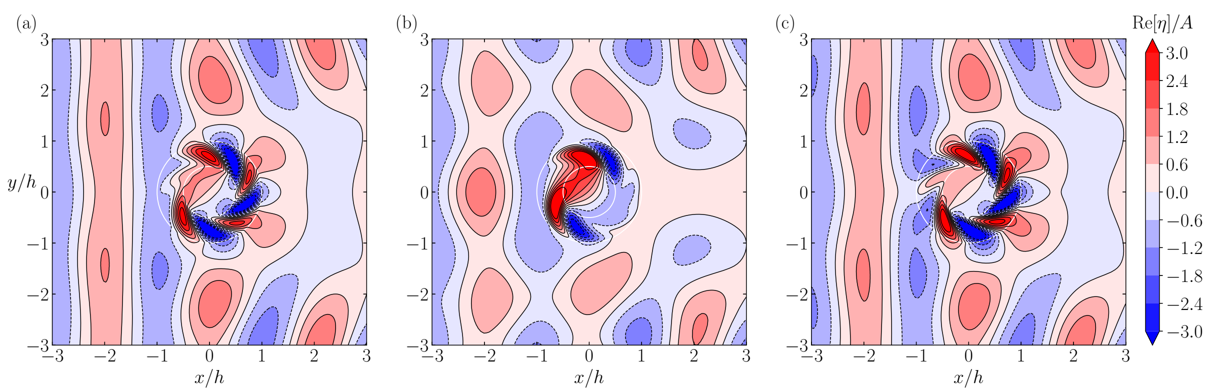

Figure 3: Instantaneous free surface at $t = 0$ for $R/h = 1.0$, $R_\mathrm{i}/R = 0.5$, $\mu(R) = \pi/4$, and $kh = \pi$: (a) Case 1 ($a_1 = 0.6617$); (b) Case 2 ($a_2/h = 0.4689$); (c) Case 3 ($a_3\sqrt{h} = 0.7877$).

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[1] S. Zheng, R. Porter, H. Liang, and D. Greaves. 2022. Water wave interaction with an annular metamaterial cylinder. *The 37th IWWWFB, Giardini Naxos, Italy.*