On interior 'high spots' in two-dimensional sloshing

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Considering the two-dimensional sloshing problem, we construct domains with 'high spots' (that is, points, where the free surface elevation is extremal for the fundamental eigenmode) located inside the surface's rest position. The notion of a high spot was introduced in [1], where a characterization of 2D domains without interior high spots was also given.

1 Statement of the problem

Let an inviscid, incompressible, heavy fluid occupy an infinite canal of uniform cross-section bounded above by a free surface of finite width. The surface tension is neglected and the fluid motion is assumed to be irrotational and of small-amplitude. The latter assumption allows us to linearize boundary conditions on the free surface which leads to the following problem in the case of the two-dimensional motion in planes normal to the generators of the canal bottom. Taking Cartesian coordinates (x, y) in the plane of motion so that the x-axis lies in the mean free surface, whereas the y-axis is directed upwards, and removing a time-harmonic factor, the following boundary value problem arises for the real-valued velocity potential $u(x, y)$:

$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \qquad u_y = \nu u \quad \text{on } F, \qquad \partial u/\partial n = 0 \quad \text{on } B. \tag{1}$$

Here the canal's cross-section W is a bounded simply connected domain, whose piecewise smooth boundary ∂W has no cusps. One of the open arcs forming ∂W is an interval F of the x-axis (the free surface of fluid at rest), and the bottom $B = \partial W \setminus \overline{F}$ is the union of open arcs, lying in the half-plane $y < 0$, complemented by corner points (if there are any) connecting these arcs. The orthogonality condition

$$\int_F u \, \mathrm{d}x = 0 \tag{2}$$

is supposed to hold, thus excluding $\nu = 0$, which is an eigenvalue of (1), and so the spectral parameter ν is equal to ω^2/g, where ω is the radian frequency of the fluid oscillations and g is the acceleration due to gravity. Properties of eigenvalues and eigenfunctions of (1), (2) are well known; see a summary in [2].

Nodal domains and their properties. Let $N(u) = \{(x, y) \in \overline{W} : u(x, y) = 0\}$ be the set of nodal lines of a sloshing eigenfunction u. A connected component of $W \setminus N$ is called a nodal domain. It follows from the first and the last conditions (1) that each nodal domain has a piecewise smooth boundary without cusps. Their properties are closely related to our considerations, and so we provide a summary of assertions proved in [2]:

(i) *If R is a nodal domain, then $\overline{R} \cap \overline{F}$ is the union of a finite number of closed subintervals of $\overline{F}$ (possibly a single interval); the eigenfunction u_n cannot change sign more than $2n$ times on F.*

(ii) *The number of nodal domains corresponding to the nth eigenfunction u_n is less than or equal to $n + 1$.*

Combining these properties and condition (2), one concludes that a fundamental sloshing eigenfunction u has a single nodal line which divides W into two nodal domains; this line has one or both ends on F.

2 Construction of domains with interior high spots (rigorous results)

We apply the semi-inverse method in which a prescribed eigenmode u satisfies the first two conditions (1) at the outset, whereas the last condition and the requirement that u is a fundamental eigenfunction are used for determining the shape of domain's bottom.

Let us consider the particular pair velocity potential/stream function, namely:

$$u(x,y) = \int_0^\infty \frac{\cos k(x-\pi) + \cos k(x+\pi)}{k-\nu} \, \mathrm{e}^{ky} \, \mathrm{d}k \,, \quad v(x,y) = \int_0^\infty \frac{\sin k(x-\pi) + \sin k(x+\pi)}{\nu - k} \, \mathrm{e}^{ky} \, \mathrm{d}k \,. \tag{3}$$

Following [3, Subsect. 4.1.1], we put $\nu = 3/2$ in which case both numerators vanish at $k = \nu = 3/2$. Hence the integrands have no singularities, and we have two usual converging infinite integrals.

It is easy to verify that u and v are conjugate harmonic functions in $\mathbb{R}^2_-$ such that

$$u(-x,y) = u(x,y) \quad \text{and} \quad v(-x,y) = -v(x,y).$$

Moreover, u and v are infinitely smooth up to $\partial\mathbb{R}^2_-$ with $\{x = \pm\pi,\, y = 0\}$ excluded, whereas $[u_y - \nu u]_{y=0}$ is equal to a linear combination of Dirac's measures at $x = \pi$ and $x = -\pi$. Therefore,

$$u_y = \nu u \quad \text{on } \partial\mathbb{R}^2_- \setminus \{x = \pm\pi,\, y = 0\}. \tag{4}$$

The calculated nodal lines of u and v are shown in Fig. 1 (b); the line plotted in solid has the following properties; see [2, Prop. 2.1] for the proof.

Proposition 1. *If $\nu = 3/2$ in (3), then along with $\{x = 0,\, y < 0\}$, there is a single nodal line of $v(x,y)$ in $\mathbb{R}^2_-$, which is smooth, symmetric about the y-axis and having both ends on the x-axis. The right one, say $(x_0, 0)$, lies between the origin and $(\pi, 0)$.*

Thus, the nodal line of v with both ends on the x-axis defines the bottom $B_{3/2}$ of a fluid domain (in Fig. 1 (b), it is denoted $W_{3/2}$), because the Cauchy–Riemann equations yield that the last condition in (1) is fulfilled on this line for u given by (3). On the free surface $F_{3/2}$ of this domain, the second condition (1) holds in view of (4), and so u satisfies the sloshing problem with $\nu = 3/2$ in $W_{3/2}$.

Furthermore, property (ii) of nodal domains implies that u is the fundamental eigenfunction corresponding to $\nu = 3/2$. Indeed, u has only one nodal line in $W_{3/2}$ (see the dashed line in Fig. 1 (b) plotted within $W_{3/2}$), which was proved in [2, Th. 2.6]; namely, we have the following assertion.

Theorem 1. *In the domain $W_{3/2}$, the sloshing eigenfunction u given by (3) with $\nu = 3/2$ has a single nodal line with endpoints $(\pm x_n, 0)$, where $x_n \in (0, x_0)$ is the only minimum point of $v(x, 0)$ on $\{x > 0\}$.*

Since $u(\pm x_n, 0) = 0$, this function has an extremum between $-x_n$ and x_n, attained at $x = 0$ in view of symmetry. Thus, we arrived at the following.

Corollary 1. *In the fluid domain $W_{3/2}$, there is an interior high spot of the fundamental sloshing eigenfunction u given by (3) with $\nu = 3/2$; it is located at the origin.*

The representation of $u(x, 0)$ valid on $[0, \pi)$ (see [2, Form. (2.9)]) implies that $u_x(0, 0) = 0$ and $u_{xx}(0, 0) < 0$. Hence $u(x, 0)$ attains maximum at the high spot $x = 0$.

There are two other interior high spots on $F_{3/2}$, which correspond to minima of $u(x, 0)$. Each of them is close to the corresponding endpoint of the free surface $F_{3/2}$. Their existence follows from

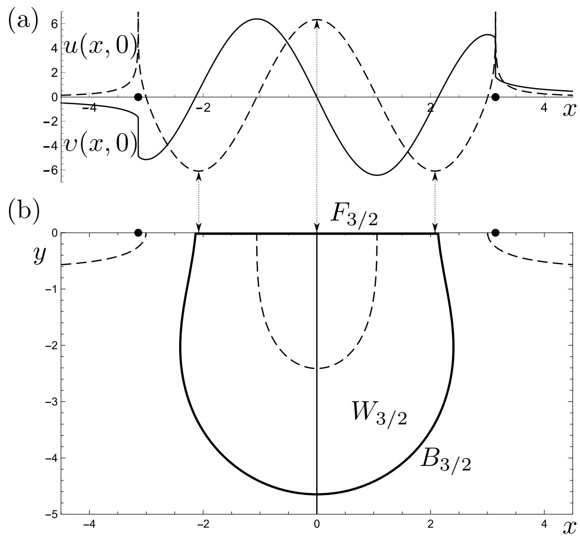

Figure 1: Plotted for $\nu = 3/2$: (a) the traces $u(x,0)$ (dashed line) and $v(x,0)$ (solid line); (b) the nodal lines of u (dashed lines) and v (solid line) given by (3). High spots on $F_{3/2}$ are marked by the arrows connecting them with the extrema of the velocity potential trace.

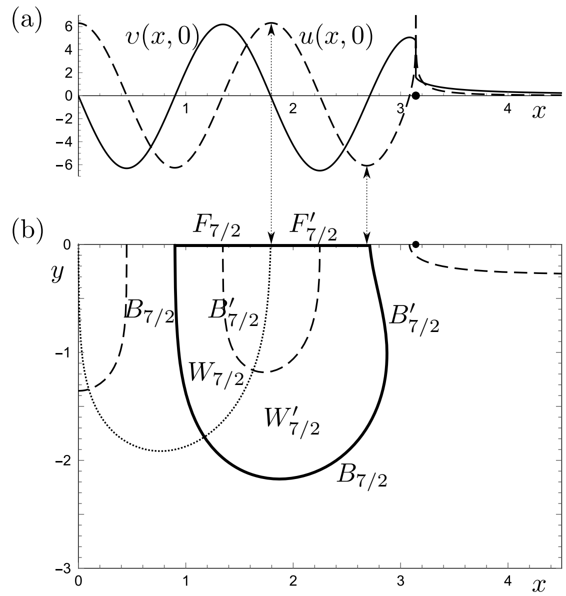

Figure 2: Plotted for $\nu = 7/2$: (a) the traces $u(x,0)$ (dashed line) and $v(x,0)$ (solid line) given by (3); (b) the nodal lines of u (dashed lines), and the level lines $v \approx -0.023145$ (solid and dotted lines). Interior high spots on $F_{7/2}$ are marked by arrows connecting them with extrema of the velocity potential trace.

the representation of $v(x,y)$ (see [2, Form. (2.7)]), implying that $v_y(x_0,0) = u_x(x_0,0) = \frac{2\,x_0}{\pi^2 - x_0^2} > 0$, and so $u_x(x_h,0) = 0$ at some $x_h < x_0$. Hence, an interior high spot is located at $(x_h,0)$ on the left of the endpoint $(x_0,0)$. By symmetry, $(-x_h,0)$ is also an interior high spot located on the right of the endpoint $(-x_0,0)$.

Computations give that $x_h \approx 2.077836$, whereas the endpoint of the free surface $F_{3/2}$ is at $(x_0,0)$ with $x_0 \approx 2.132704$, that is, the distance from the high spot to the endpoint is approximately 0.054868, which is less than 3 % of of the distance from the origin to the endpoint of $F_{3/2}$.

Since the negative y-axis is the nodal line of v with $\nu = 3/2$, the half-domain on the right (left) of this axis provides an example of domain with a single interior high spot.

There is another property which is quite evident for the geometry of $W_{3/2}$ shown in Fig. 1 (b).

Definition 1. A fluid domain W satisfies John's condition if it is confined to the strip bounded by the straight vertical lines through the endpoints of the free surface F. Domains violating this condition are called bulbous.

Proposition 2. *The domain $W_{3/2}$ is bulbous.*

This follows by demonstrating that $y'(x_0) < 0$ for the implicit function $x \mapsto y$ defined by the equation $v(x,y) = 0$ in a neighbourhood of $(x_0,0)$ — the right endpoint of $B_{3/2}$.

Rigorous considerations, analogous to those above, are applicable when u and v are given by (3) with $\nu = 5/2$.

3 Further examples of domains with interior high spots (numerical results)

Here we present another domain with interior high spots obtained numerically by virtue of the following procedure. For a specified ν, an appropriate nonzero level line of v is chosen to define the bottom. The criterion for choosing the required level is that the corresponding line has two branches crossing transversally at a stagnation point; one of these branches (or both) serves as the bottom.

For $\nu = 7/2$ the stagnation point of v occurs at the level approximately equal to -0.023145; it is the point of intersection of the solid and dotted lines in Fig. 2 (b). The solid line encloses the domain $W_{7/2}$ divided into two nodal domains by the dashed line (a nodal line of u), and so u is the fundamental eigenfunction corresponding to $\nu = 7/2$ in $W_{7/2}$.

There are two interior high spots on $F_{7/2}$: near its middle at $x \approx 1.795807$, and close to the right endpoint of $F_{7/2}$ at $x \approx 2.685549$, within approximately 0.026076 from the endpoint.

Furthermore, u given by (3) with $\nu = 7/2$ is the fundamental eigenfunction corresponding to this ν in another domain, say $W'_{7/2} \subset W_{7/2}$, which is confined between the solid and dotted lines to the right of the latter one; see Fig. 2 (b). Indeed, $v \approx -0.023145$ on both these lines, and there are two nodal domains in $W'_{7/2}$ separated by the dashed line. There are two interior high spots on $F'_{7/2} \subset F_{7/2}$, and both of them are close to the endpoints of $F'_{7/2}$, the left of which is at $x \approx 1.789875$, and so the high spot located near the middle of $F_{7/2}$ is inside of $F'_{7/2}$. Moreover, $W'_{7/2}$ is bulbous on both sides which distinguishes it from $W_{7/2}$; see Fig. 2 (b).

Also, the same u is the fundamental eigenfunction corresponding to $\nu = 7/2$ in the domain $W_{7/2} \setminus \overline{W'_{7/2}}$, on the left of the dotted line in Fig. 2 (b). This domain satisfies John's condition and has no interior high spots on its free surface.

In conclusion, we notice that similar nodal domains with interior high spots are found for $\nu = 2, 3$ provided u and v are analogous to (3) but having opposite parity with respect to x.

4 Concluding remarks

Here are some characteristic features of the found domains with interior high spots: (I) many of these domains, but not all, have multiple interior high spots; (II) every such domain has at least one interior high spot located close to an endpoint of the free surface; (III) these domains are bulbous on the side, where an interior high spot is located close to an endpoint of the free surface; (IV) in domains with a single interior high spot, the nodal line of the velocity potential connects the free surface and the bottom; (V) in domains with multiple interior high spots, both types of potential nodal lines are possible: connecting the free surface and the bottom and connecting two different points on the free surface.

Troughs. It is clear that a sloshing domain $W \subset \mathbb{R}^2_-$ defines a trough $W \times (0, \ell) \subset \mathbb{R}^3_-$ of any length $\ell > 0$. Moreover, if $u(x, y)$ is a fundamental eigenmode of sloshing in W, then this function plays the same role for $W \times (0, \ell)$. Therefore, if W has an interior high spot, then there is a straight line (parallel to trough's generators) in the free surface $F \times (0, \ell)$, each point of which is a high spot interior with respect to this free surface.

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