Water entry of a conical shell

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The axisymmetric problem of a conical shell impact onto an inviscid and incompressible liquid of infinite depth is considered. The problem is formulated in cylindrical coordinates z, r, where z is the vertical coordinate and r is the radial coordinate. Initially, $t = 0$, the liquid occupies the lower half-space $z < 0$ and the position of the shell is described by the equation $z = r \tan\beta$, $r < R\cos\beta$, where β is the deadrise angle of the cone, and R is the length of the cone generator (Leissa, 1973). The shell is not stressed before impact. The cone touches the flat and horizontal surface of the liquid at a single point taken as the origin of the coordinate system, see figure 1. The speed of the shell V is constant. The flow caused by the elastic cone impact is assumed potential and axisymmetric. The deadrise angle of the cone β is small, which makes it possible to approximate the hydrodynamic loads acting on the impacting conical shell using the Wagner approach. Gravity, surface tension and viscous effects are not taken into account. Within the Wagner approach only the early stage of the impact is considered, during which the vertical displacement of the cone is comparable with the vertical dimension of the cone, $R\sin\beta$, and is much smaller than the horizontal dimension of the wetted part of the cone, which is of the order of $R\cos\beta$. During the early stage of impact, the flow region is approximated by the lower half-space $z < 0$, the boundary conditions on the free surface, $z = \eta(r,t)$, $r > a(t)$, and on the wetted part of the deformed cone, which is approximated as $z = r\tan\beta - w(x,t)$, $r < a(t)$, are linearised and imposed on the plane $z = 0$. Here $x = r/\cos\beta$ is the distance along the cone generator measured from the tip of the cone, $a(t)$ is the radius of the wetted part of the deformed cone during the impact stage, when $a(t) < R\cos\beta$, and $w(x,t)$ is the normal elastic displacement of the shell elements caused by the impact loads. During the next, penetration, stage, which starts when the cone is completely wetted but continues to penetrate into the liquid, we set $a(t) = R\cos\beta$.

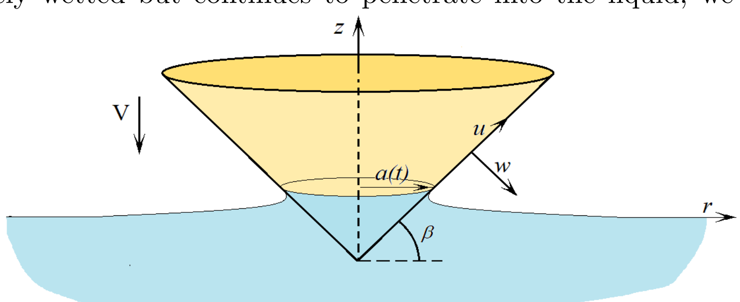

Fig. 1 Vertical impact of a conical shell onto liquidof infinite depth.

Within the Wagner approach, the hydrodynamic pressure is given by the linearised Bernoulli equation, and the boundary-value problem for the velocity potential $\varphi(r,z,t)$ reads (see Scolan, 2004),

$$\nabla^2\varphi = 0 \quad (z<0), \qquad \varphi = 0 \quad (z=0, r>a(t)), \qquad \varphi_z = -V - w_t(r,t) \quad (z=0, r<a(t)), \quad (1)$$

$$p(r,z,t) = -\rho\varphi_t(r,z,t),$$

where ρ is the liquid density. The elastic displacements in the normal, $w(x,t)$, and tangential, $u(x,t)$, directions with respect to the initial shape of the cone, see figure 1, are governed by the most complete linear equations of the conical shell, see Soedel (2004) and Leissa (1973),

$$\rho_s h w_{tt} + D\nabla^4 w + \frac{12D\tan^2\beta}{x^2h^2}\Big(w + \tan\alpha\big[u + \nu x u_x\big]\Big) = P_{ext}(x,t), \quad (2)$$

$$\rho_s h u_{tt} - K\Big(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u\Big) - \frac{K\tan\beta}{x^2}\big[\nu x w_x - w\big] = 0, \tag{3}$$

$$K = \frac{12}{h^2}D, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \qquad \tan\alpha = \frac{1}{\tan\beta}, \qquad (0 < x < R)$$

with the edge,

$$w = w_x = u = 0 \qquad (x = 0, R), \tag{4}$$

and initial,

$$w(x,0) = w_t(x,0) = u(x,0) = u_t(x,0) = 0, \tag{5}$$

conditions. Here ρ_s, E and ν are the density, Young module and Poisson ratio of the shell material, h is the shell thickness, and $P_{ext}(x,t) = -p(x,0,t)$ is the external normal load acting on the shell. Note that we take $x \approx r$ in (1) and (2) for small deadrise angle β.

The problem (1)-(5) was solved by Scolan (2004) neglecting the tangential displacement $u(x,t)$ and the third term on the left-hand side of shell equation (2). This approximation implies that the conical shell was modeled as a circular elastic plate clamped at the edge $r = R$ and at the centre $r = 0$, which corresponds to the tip of the cone. The hydrodynamic part (1) was solved without further simplifications. Grigolyuk and Gorshkov (1971) solved the problem using a nonlinear theory of conical shells approximating the normal displacement $w(x,t)$ by a quadratic function, the contact radius $a(t)$ was approximated by that for a rigid cone supported by an equivalent spring. Inertia of the shell in the tangential direction and the hydrodynamic loads in this direction were neglected. Malenica et al (2022) numerically investigated the DAF for the conical shell using the approximation of the structure by Scolan (2004).

The problem (1)-(5) in the dimensionless variables,

$$r = R\tilde{r}, \quad z = R\tilde{z}, \quad t = T_{sc}\tilde{t}, \quad w = VT_{sc}\tilde{w}(\tilde{r},\tilde{t}), \quad u = VT_{sc}\tan\beta\,\tilde{u}(\tilde{r},\tilde{t}), \quad \varphi = VR\tilde{\varphi}(\tilde{r},\tilde{z},\ t),$$

$$p = \rho V^2 \tilde{p}(\tilde{r},\tilde{z},\tilde{t})/\sin\beta, \quad a = R\tilde{a}, \quad T_{sc} = R\sin\beta/V,$$

takes the form (a tilde is dropped below),

$$\nabla^2\varphi = 0 \ \ (z<0), \quad \varphi(r,0,t) = 0 \quad (r > a(t)), \quad \varphi_z(r,0,t) = -1 - w_t(r,t) \quad (r < a(t)), \tag{6}$$

$$\alpha w_{tt} + \nabla^4 w + \frac{\delta}{r^2}\big(w + \nu r u_r + u\big) = -\alpha\gamma\varphi_t(r,0,t), \tag{7}$$

$$u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + \frac{1}{r^2}\big(\nu r w_r - w\big) = \varepsilon\alpha u_{tt}, \tag{8}$$

$$w = w_x = u = 0 \qquad (x = 0, 1), \qquad w = w_t = u = u_t = 0 \qquad (t = 0). \tag{9}$$

Here

$$\alpha = 12(1-\nu^2)\Big(\frac{VR}{c_p h\sin\beta}\Big)^2, \quad c_p = \sqrt{\frac{E}{\rho_s}}, \quad \gamma = \frac{\rho R}{\rho_s h}, \quad \delta = 12\tan^2\beta\frac{R^2}{h^2}, \quad \varepsilon = \frac{1}{12}\Big(\frac{h}{R}\Big)^2.$$

We assume that the impact conditions are such that $\alpha = O(1)$. For shells, $\varepsilon \ll 1$ and $\gamma < 1$. As to the parameter δ, we have $\tan\beta \ll 1$ and $R/h \gg 1$ with their product can be any. Therefore, we can neglect the term $\varepsilon\alpha u_{tt}$, as it was done by Grigolyuk and Gorshkov (1971) but should keep the term with δ. In the study by Scolan (2004), the term with δ was neglected. For the parameters used by Malenica et al. (2022), $E = 2.1\times10^{11}$Pa, $\rho_s = 7850$ kg/m^3, $h = 0.01$ m, $R = 1$ m, $\beta = 10$ deg, and $\nu = 0.3$, $\rho = 1000$ kg/m^3 and $V = 2$ m/s, we calculate $\delta = 3730.9445$, $\varepsilon = 8.33\times10^{-6}$, $c_p = 5172.2$ m/s, $\gamma = 12.74$, $\alpha = 0.5412$, $\alpha\gamma = 6.895$.

The coupled problem of hydroelasticity (6) can be solved by the normal mode method using the dry modes of the conical shell. However, we are not sure that the dry modes of the conical shell are orthogonal, and it would be difficult to normalise them if they are orthogonal. It is suggested to use the dry modes as in Scolan (2004), which are the modes of the circular plate clamped at the centre and at the edge. These dry modes correspond to the shell equation (7), where $\delta = 0$. The Scolan's modes are solutions of the eigen-value problem,

$$\nabla^4 \psi_n = \lambda_n^4 w_n \qquad (0 < r < 1), \qquad \psi_n = \psi_n' = 0 \qquad (r = 0, 1). \tag{10}$$

The modes are orthogonal,

$$\int_0^1 \psi_n(r)\psi_m(r) r \mathrm{d}r = \delta_{nm}, \tag{11}$$

and have the form

$$\psi_n(r) = A_n \Big\{ C_n \big[J_0(\lambda_n r) - I_0(\lambda_n r) \big] - Y_0(\lambda_n r) - \frac{2}{\pi} K_0(\lambda_n r) \Big\}, \quad C_n = \frac{Y_0(\lambda_n) + (2/\pi)K_0(\lambda_n)}{J_0(\lambda_n) - I_0(\lambda_n)}, \tag{12}$$

see Scolan (2004) for the details and the dispersion relation for the eigen values λ_n.

The normal deflection of the conical shell and the velocity potential are sought in the form of the series,

$$w(r,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(r), \quad \varphi(r,z,t) = \varphi_0(r,z,a) + \sum_{n=1}^{\infty} \dot{a}_n(t)\varphi_n(r,z,a), \tag{13}$$

$$q(r,t) = w_t(r,t) + \gamma\varphi(r,0,t), \quad q(r,t) = \sum_{n=1}^{\infty} q_n(t)\psi_n(r), \tag{14}$$

where the potentials $\varphi_n(r,z,a)$ are the solutions of the same boundary problem as (6) but with the conditions $\varphi_{0z}(r,0,a) = -1$, $\varphi_{nz}(r,0,a) = \psi_n(r)$, where $r < a(t)$. Solving (8) with respect to $u(r,t)$ for $\varepsilon = 0$, substituting the result and series (12), (13) in equation (7), multiplying both sides by $r\psi_m(r)$, where $m \geq 1$, integrating in r from 0 to 1, and using (10) and (11), we find

$$q_m = \dot{a}_m(t) + \gamma \sum_{n=1}^{\infty} \dot{a}_n(t) S_{nm}(a) + \gamma f_m(a), \tag{15}$$

$$f_m(a) = \int_0^a \varphi_0(r,0,a)\psi_m(r) r \mathrm{d}r, \qquad S_{nm}(a) = \int_0^a \varphi_n(r,0,a)\psi_m(r) r \mathrm{d}r, \tag{16}$$

$$\alpha \dot{q}_m + \lambda_m^4 a_m + \delta \sum_{n=1}^{\infty} a_n(t) T_{mn} = 0. \tag{17}$$

Here the symmetric matrix $\mathbb{S}(a)$ with the elements $S_{nm}(a)$ is known as the added-mass matrix of the modes $\psi_n(r)$. The matrix $\mathbb{T}$ with the elements T_{nm} is symmetric and depends only on the Poisson ratio ν,

$$T_{mn} = \frac{1}{2}(1+\nu)^2 \int_0^1 \psi_n(r)\mathrm{d}r \int_0^1 \psi_m(r)\mathrm{d}r -$$

$$(1-\nu^2)\Big\{ \int_0^1 \psi_m(r)\psi_n(r)\frac{\mathrm{d}r}{r} - \frac{1}{2}\int_0^1 \psi_m(r) \int_0^r \psi_n(r_0)\mathrm{d}r_0 \frac{\mathrm{d}r}{r^2} - \frac{1}{2}\int_0^1 \psi_n(r) \int_0^r \psi_m(r_0)\mathrm{d}r_0 \frac{\mathrm{d}r}{r^2} \Big\}.$$

The matrix $\mathbb{T}$ is precalculated numerically using orthonormal functions (12). The added-mass matrix $\mathbb{S}(a)$ is the same as in Scolan (2004). It should be calculated numerically at each step of integration of (15) and (17) in time, which is for $0 < a \leq 1$.

Equations (15) and (17), where $m \geq 1$, can be written in the matrix form,

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (\mathbb{I} + \gamma\mathbb{S})^{-1}(\vec{q} - \gamma\vec{f}(a)) \qquad \frac{\mathrm{d}\vec{q}}{\mathrm{d}t} = -\frac{1}{\alpha}(\mathbb{D} + \delta\mathbb{T})\vec{a}, \quad \vec{a}(0) = 0, \quad \vec{q}(0) = 0, \tag{18}$$

where $\mathbb{D}$ is a diagonal matrix, $\mathbb{D} = \mathrm{diag}(\lambda_1^4, \lambda_2^4, \lambda_3^4, ...)$, $\vec{a}(t) = (a_1, a_2, a_3, ...)^T$, $\vec{q}(t) = (q_1, q_2, q_3, ...)^T$, $\vec{f}(a) = (f_1, f_2, f_3, ...)^T$,

$$f_m(a) = -\frac{2}{\pi}a^3 \int_0^{\frac{\pi}{2}} \psi_m(a\sin\alpha)\cos^2\alpha\sin\alpha\mathrm{d}\alpha, \quad S_{nm}(a) = -\frac{2}{\pi}a^3 \int_0^1 x^2 Q_n(ax)Q_m(ax)\mathrm{d}x,$$

$$Q_n(x) = \int_0^{\frac{\pi}{2}} \psi_n(a\sin\alpha)\sin\alpha\mathrm{d}\alpha. \tag{19}$$

The integrals in (19) are evaluated numerically. The radius $a(t)$ of the contact region of the conical shell satisfies the equation

$$a(t) = \frac{4}{\pi}\Big(t + \sum_{n=1}^{\infty} a_n(t)Q_n(a)\Big). \tag{20}$$

The system (18), (20) is integrated in time up to $a(t_*) = 1$. For $t > t_*$, when the shell is completely wetted but continues penetration into the liquid, we set $a = 1$, $\vec{f} = 0$ and continue the numerical integration in time.

Within the decoupled quasi-static approximation, we neglect $w_t(r,t)$ in the equations of the shell dynamics and $w(r,t)$ in (20). Then, equation (15) gives $q_m = \gamma f_m(a)$ and (17) provides the coefficients $a_n(t)$ in an explicit form,

$$\vec{a} = \frac{128}{\pi^4}t^2\alpha\gamma(\mathbb{D} + \delta\mathbb{T})^{-1}\vec{Q}(\frac{4t}{\pi}) \quad (0 < t < \pi/4), \qquad \vec{a} = 0 \quad (t > \pi/4).$$

Within the original and quasi-static theories, the physical strains on the inner and outer surfaces of the conical shell are given by

$$\varepsilon_{xx} = \varepsilon_{sc}\left(\sqrt{\frac{\delta}{3}}\frac{\partial u}{\partial x} \pm \frac{\partial^2 w}{\partial x^2}\right), \qquad \varepsilon_{\theta\theta} = \varepsilon_{sc}\frac{1}{x}\left(\sqrt{\frac{\delta}{3}}(u + w) \pm \frac{\partial w}{\partial x}\right), \qquad \varepsilon_{sc} = \frac{h}{2R}\sin\beta,$$

where $u(x,t)$, $w(x,t)$ and x are dimensionless variables.

The displacements and strains computed by the original and quasi-static decoupled theories will be compared and presented at the workshop together with the DAF of conical shells.

References

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