Wave scattering by a large compact array of thin vertical plates

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1. INTRODUCTION

In [1], two different models were presented to determine two-dimensional wave propagation through a finite periodic array of vertical surface-piercing barriers. One method is exact and based on the Bloch-Floquet theory; the other method is approximate and replaces the barrier array with an effective medium using the homogenization method. Under the wide-spacing approximation, [2] developed a recursive model to solve for the reflection and transmission of waves over a three-dimensional array of vertical plates. In this paper, we extend the methods of [1] to investigate wave scattering by a plate array similar to that considered by [2], the principal difference being that the spacing between adjacent plates is assumed small.

2. PERIODIC PLATE PROBLEM

Consider an array of thin fixed vertical plates having the same width c and extending uniformly throughout the water depth h. The array of paddles is periodically distributed in the y direction with a periodicity $d = a + c$, a being the spacing between two adjacent plates, while in the x direction there are $N + 1$ rows of parallel paddles with identical separation b. Three-dimensional Cartesian coordinates (x, y, z) are defined such that the plates occupy $\{x = nb, a + md < y < d + md, -h < z < 0\}$ $(n = 0, 1, \cdots, N, m \in \mathbb{Z})$. A plane wave with the angular frequency ω is incident at an angle θ relative to the positive x-axis.

Following the method of [1] we can solve wave scattering by multiple rows of plates by first determining solutions relating to infinite rows which allows us to focus on a single periodic cell defined by the cross-section $D = \{0 < x < b, 0 < y < d\}$ based on seeking Bloch-Floquet modes. Since the geometry is uniform in the depth, the solution can be expressed in terms of the two-dimensional spatial potential $\phi(x, y)$ satisfying

$$\left(\nabla^2 + k^2\right) \phi(x, y) = 0, \qquad \text{in } D, \tag{1}$$

where the wavenumber, k, is the positive real root of $K = \omega^2/g = k \tanh kh$. The no-flow condition on the plates requires

$$\phi_x(0, y) = \phi_x(b, y) = 0, \quad \text{for } a < y < d. \tag{2}$$

Due to periodicity, on the fluid interfaces ϕ satisfies Bloch-Floquet conditions

$$\phi(x, d) = \mathrm{e}^{\mathrm{i}\alpha d}\phi(x, 0), \quad \text{and} \quad \phi_y(x, d) = \mathrm{e}^{\mathrm{i}\alpha d}\phi_y(x, 0), \quad \text{for } 0 < x < b, \tag{3}$$

$$\phi(b, y) = \mathrm{e}^{\mathrm{i}\beta b}\phi(0, y), \quad \text{and} \quad \phi_x(b, y) = \mathrm{e}^{\mathrm{i}\beta b}\phi_x(0, y) \equiv \mathrm{e}^{\mathrm{i}\beta b}U(y), \quad \text{for } 0 < y < a, \tag{4}$$

where $\alpha = k \sin\theta$ is fixed by the frequency and incident wave angle and β is the Bloch wavenumber to be determined; $U(y)$ is defined as the horizontal velocity at $x = 0$. [1] has illustrated that we only need to consider the real values of $\beta \in (0, \pi/b]$ and the complex values of $\beta = \mathrm{i}\hat{\beta}_0$ and $\beta = \pi/b + \mathrm{i}\hat{\beta}_1$ (where $\hat{\beta}_0$, $\hat{\beta}_1 \in \mathbb{R}$).

In D, the general solution of $\phi(x, y)$ satisfying (1) and (3) can be expressed as

$$\phi = \sum_{n=-\infty}^{\infty} \left(a_n \mathrm{e}^{\mathrm{i}\gamma_n x} + b_n \mathrm{e}^{-\mathrm{i}\gamma_n x}\right) \mathrm{e}^{\mathrm{i}\alpha_n y}, \quad \text{with} \quad \gamma_n = \left\{ \begin{array}{ll} \sqrt{k^2-\alpha_n^2}, & k \geq |\alpha_n|, \\ \mathrm{i}\sqrt{\alpha_n^2-k^2}, & k < |\alpha_n|, \end{array} \right. \tag{5}$$

where $\alpha_n = \alpha + 2n\pi/d$ and a_n and b_n for $n \in \mathbb{Z}$ are undetermined coefficients. After substituting (5) into the velocity periodic condition in (4) and using the orthogonality of the functions $\mathrm{e}^{\mathrm{i}\alpha_n y}$ over $0 < y < d$, a_n and b_n can be expressed in terms of integrals with $U(y)$. Then, applying the pressure periodic condition in (4) to (5) results in

$$\sum_{n=-\infty}^{\infty} \frac{\cos(\beta b) - \cos(\gamma_n b)}{\gamma_n d \sin(\gamma_n b)} \mathrm{e}^{\mathrm{i}\alpha_n y} \int_0^a \mathrm{e}^{-\mathrm{i}\alpha_n y'} U(y') \mathrm{d}y' = 0. \tag{6}$$

Since the analysis of flow close to the edge of the barrier reveals the velocity behaves as the inverse square root of the distance to the edge, $U(y)$ can be expanded as (see [3])

$$U(y) \approx \sum_{p=0}^{P} w_p u_p(y), \quad \text{with} \quad u_p(y) = \frac{T_p(2y/a-1)}{\pi\sqrt{(a/2)^2 - (y-a/2)^2}}, \tag{7}$$

where P is a numerical truncation parameter and $T_p(\cdot)$ is a Chebyshev polynomial. Substituting approximation (7) into (6), multiply through by $u_q(y)$ and integrating over $0 < y < a$ leads to the following homogeneous system of equations:

$$\sum_{p=0}^{P} w_p (-\mathrm{i})^p \mathrm{i}^q \sum_{n=-\infty}^{\infty} \frac{\cos(\beta b) - \cos(\gamma_n b)}{\gamma_n d \sin(\gamma_n b)} J_p(\alpha_n a/2) J_q(\alpha_n a/2) = 0, \quad \text{for } q = 0, 1, \cdots, P \tag{8}$$

where $J_p(\cdot)$ is the pth order Bessel function the first kind. In order to determine the real values of $\beta \in (0, \pi/b]$, $\hat{\beta}_0 > 0$ and $\hat{\beta}_1 > 0$, we require (8) has a non-trivial solution. It can be shown that the determinant of the matrix formed by elements in (8) is always real for the values of β under consideration such that a standard root finding method can be applied.

Next, we present a useful orthogonality relation which is extensively used in the following. For convenience, we label eigenvalues $\beta = \pm\beta^{(k)}$ ($k = 0, 1, 2, \ldots$) where the numbering system is that ascending real values take precedence over complex values, ordered by their increasing imaginary part. Each eigenvalue $\pm\beta^{(k)}$ is associated with a corresponding eigenfunction which is labelled as $\phi^{(\pm k,+)}$. Besides, we have another two eigenfunctions $\phi^{(\pm k,-)}$ satisfying $\phi^{(\pm k,-)}(x,y) = \phi^{(\pm k,+)}(x, a-y)$. After applying Green's second identity to $\phi^{(+k,+)}$ and $\phi^{(\pm j,-)}$ in D, as long as $\beta^{(k)} \neq \pi/b$ (which relates to the standing waves) we can determine

$$\left\{ \begin{array}{l} \displaystyle\int_0^a \left[\phi^{(+k,+)}(0,y)\phi_x^{(+j,-)}(0,y) - \phi^{(+j,-)}(0,y)\phi_x^{(+k,+)}(0,y)\right] \mathrm{d}y = 0, \\ \displaystyle\int_0^a \left[\phi^{(+k,+)}(0,y)\phi_x^{(-j,-)}(0,y) - \phi^{(-j,-)}(0,y)\phi_x^{(+k,+)}(0,y)\right] \mathrm{d}y = E^{(+k)}\delta_{kj}, \end{array} \right. \tag{9}$$

where $E^{(+k)}$ is a scaling factor.

Then, we consider the reflection and transmission of $N+1$ rows of parallel plates which is a finite section of the problem given above. The general solution in each period $(n-1)b < x < nb$ can be expressed by the superposition of the eigenfunctions $\phi^{(\pm k,+)}$:

$$\phi_n = \sum_{k=0}^{\infty} \left[A_n^{(k)} \phi^{(+k,+)}(x-(n+1)b, y) + B_n^{(k)} \phi^{(-k,+)}(x-(n+1)b, y)\right]. \tag{10}$$

After applying orthogonality relation (9) to ϕ_n with $\phi^{(\pm j,-)}$ and ϕ_{n+1} with $\phi^{(\pm j,-)}$ and using the matching conditions for ϕ_n and ϕ_{n+1} at their fluid interfaces, it can be shown that $A_{n+1}^{(k)} = \mathrm{e}^{\mathrm{i}\beta^{(k)}b} A_n^{(k)}$ and $B_{n+1}^{(k)} = \mathrm{e}^{-\mathrm{i}\beta^{(k)}b} B_n^{(k)}$. This implies that, for any n, ϕ_n can be expressed by a single set of coefficients $A_1^{(k)}$ and $B_1^{(k)}$ relating to the first period. In the outer regions $x < 0$ and $x > Nb$, the velocity potential can be respectively written as

$$\phi_0 = \mathrm{e}^{\mathrm{i}(\gamma_0+\alpha_0 y)} + \sum_{n=-\infty}^{\infty} r_n \mathrm{e}^{\mathrm{i}(-\gamma_n x+\alpha_n y)} \quad \text{and} \quad \phi_{N+1} = \sum_{n=-\infty}^{\infty} t_n \mathrm{e}^{\mathrm{i}(\gamma_n(x-Nb)+\alpha_n y)}. \tag{11}$$

The coefficients r_n, t_n are determined by matching the velocity potential across $x = 0$ and $x = Nb$. After using the orthogonality relation (9) and decomposing the problem into a symmetric (s) and anti-symmetric (a) problem, we obtain a pair of scalar integral equations

$$\int_0^a \left[\sum_{n=-\infty}^{\infty} \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha_n y}}{\gamma_n d} H_{nk} + \begin{pmatrix} \mathrm{i}\tan(\beta^{(k)}b/2) \\ -\mathrm{i}\cot(\beta^{(k)}b/2) \end{pmatrix} \phi^{(+k,-)}(0,y) \right] \begin{pmatrix} U^s(y) \\ U^a(y) \end{pmatrix} \mathrm{d}y = -2H_{0k}, \tag{12}$$

where

$$H_{nk} = \int_0^a \mathrm{e}^{\mathrm{i}\alpha_n y} \frac{\partial}{\partial x} \phi^{(+k,-)}(0,y)\mathrm{d}y. \tag{13}$$

In the above, $[U^s(y) \pm U^a(y)]/2$ represent the horizontal velocity across the interface $x = 0$ and $x = Nb$ respectively, which can be approximated by the same expansions used in (7). Thus, the integral equation (12) can be solved using the identical method indicated above. After this, the total reflected and transmitted energy can be determined by

$$|R_N|^2 = \sum_{n=-r}^{s} \frac{\gamma_n}{\gamma_0} |r_n|^2 \quad \text{and} \quad |T_N|^2 = \sum_{n=-r}^{s} \frac{\gamma_n}{\gamma_0} |t_n|^2, \tag{14}$$

where $r = [kd(1+\sin\theta)/2\pi]$, $s = [kd(1-\sin\theta)/2\pi]$ and $[\cdot]$ denotes the integer part.

3. HOMOGENIZATION METHOD

In this section, we assume the separation between two adjacent plates is small compared with the wavelength, i.e. $O(kb) \ll 1$. We also assume that $O(kc) \sim O(kh) \sim 1$. After applying a multiscale analysis, it can be shown that the leading-order velocity potential in $0 < x < Nb$ satisfies

$$\begin{cases} (\nabla^2 + k^2)\,\phi(x,y) = 0, & 0 < y < a, \\ (\partial^2/\partial y^2 + k^2)\,\phi(x,y) = 0, & a < y < d. \end{cases} \tag{15}$$

The general solution of (15) can be expressed as

$$\phi(x,y) = \begin{cases} \mathrm{e}^{\mathrm{i}\mu x}(A\mathrm{e}^{\mathrm{i}\nu y} + B\mathrm{e}^{-\mathrm{i}\nu y}), & 0 < y < a, \\ C(x)\mathrm{e}^{\mathrm{i}ky} + D(x)\mathrm{e}^{-\mathrm{i}ky}, & a < y < d. \end{cases} \tag{16}$$

where $\mu^2 + \nu^2 = k^2$. Applying the periodicity conditions (3) and continuity of pressure and velocity across $y = a$ to (16) finally results in

$$2k\nu[\cos(kc)\cos(\nu a) - \cos(\alpha d)] = (k^2 + \nu^2)\sin(kc)\sin(\nu a). \tag{17}$$

There exist a series of real values of μ and pure imaginary values of $\mu = \mathrm{i}\hat{\mu}$ satisfying (17).

Now consider that the region $0 < x < Nb$ is occupied by the effective medium described in above. Thus the velocity potential in this region can be approximated by

$$\phi(x,y) = \sum_{n=0}^{\infty}(a_n e^{i\mu_n x} + b_n e^{-i\mu_n x})(A_n e^{i\nu_n y} + B_n e^{-i\nu_n y}), \quad \text{for } 0 < y < a, \tag{18}$$

where ν_n are the roots of (17), $\mu_n^2 = k^2 - \nu_n^2$ and A_n and B_n are the corresponding eigensolutions. This approximate solution is completed by matching to (11) and (18) and following the Galerkin method given in [1].

4. RESULTS

Fig. 1(a-c) show the variation of the Bloch wavenumbers β and roots μ from homogenisation against non-dimensional wavenumber Kh with $a/c = 0.5$, $h/c = 1.0$ and $\theta = \pi/4$. As the frequency increases, the real values of β will increase until $\beta = \pi/b$ while μ tends to infinity. The real values of β then move into the complex plane, moving along the half infinite line $\beta = \pi/b + i\hat{\beta}_1$. The pure imaginary values of β decrease to zero before moving along the real axis. Fig. 1(d) shows the corresponding reflection coefficients when $Nb/c = 1.0$. As the spacing b/c decreases, the results from the exact model gradually approach those from the effective medium except when the real Bloch wavenumber β is close to π/b.

The fixed plates can be replaced by the paddles hinged at the sea bed and attached to the damper and spring as a system for harnessing wave energy. This requires that the no-flow condition (2) is replaced by a condition determined by combining dynamic and kinematic conditions on the paddles. Results for this problem will be presented at the Workshop.

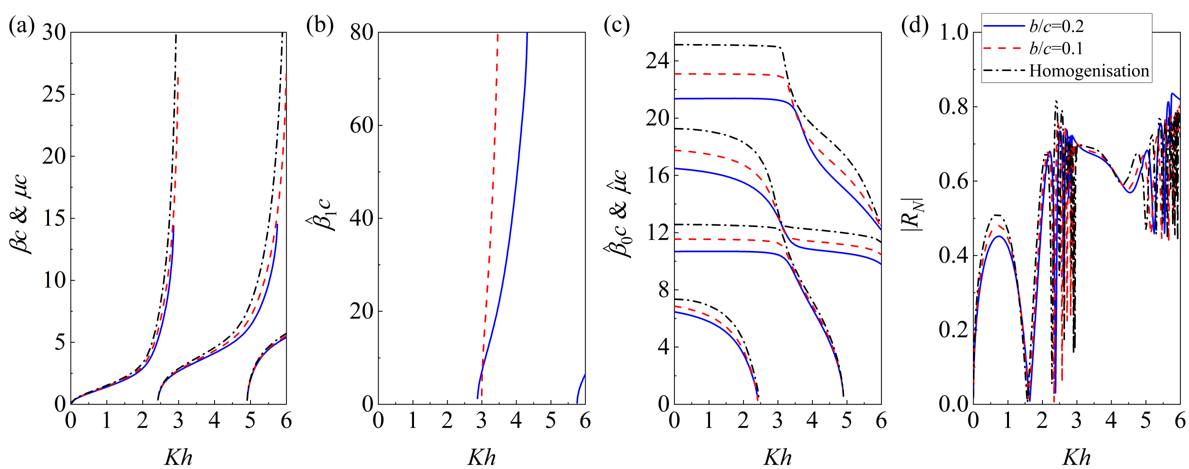

Figure 1: Variation of: (a-c) β and μ and (d) $|R_N|$ against Kh with $a/c = 0.5$, $h/c = 1.0$, $\theta = \pi/4$ and $Nb/c = 1.0$.

REFERENCES

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