Time-domain simulation of surface waves in a compressible ocean due to the motion of a circular portion of the ocean floor

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HIGHLIGHTS The impact of static compression of ocean water on surface wave generation due to a rise in the circular portion of the ocean floor is considered. The region-wise depth-dependant functions are obtained using separation of variables method. the analytical solution is obtained with the help of eigenfunction expansion method after applying the matching conditions and a newly defined inner products between the depth-dependent functions.

1 INTRODUCTION

The generation of surface ocean waves due to ocean floor movement is of paramount importance due to the possible generation of tsunami waves whose devastating impact on the coastal areas has been witnessed throughout history. The destruction caused by the 2004 Indian Ocean (Sumatra earthquake), the 2011 Tohoku Oki, 2018 Sulawesi, and Palu tsunamis are some of the contemporary ones. Consequently, the development of a model accurate enough to simulate the motion of such a surface wave kept on attracting the minds of researchers all over the world for over a century, if not more. Amongst all the different types of models developed over the years, the ones incorporating ocean water compressibility provide more accurate surface wave propagation compared to the ones that neglect such compressibility. Detailed progress on this aspect can be found in literature (1; 2; 3; 4; 5). We consider the motion of a circular portion of the ocean floor and find out the the analytical expression for the surface profile when the ocean water is considered compressible.

1.1 Mathematical formulation

We consider free-surface gravity wave propagation due to vertical movement of a circular portion area of radius a in a compressible ocean of finite depth h. The respective physical problem is formulated in a three-dimensional polar coordinate system having z- axis pointing upwards and perpendicular to the (r, θ) plane. The ocean bed is characterized as rigid and the portion of the oscillating bottom lies in $r < a$. A wave propagation due to the ocean floor disturbance is realized towards the radial direction under the assumption of linearised water wave theory. The flow is considered irrotational. We are interested in calculating the the time-dependent motion of the fluid due to a movement of the seafloor, simulating the generation of a Tsunami in three dimensions. If the portion of the ocean bed grows at an arbitrary rate $l(t)$ in the time interval $(0, \tau)$, the ocean bottom profile will be given by

$$\hat{h}(r, \theta, -h, t) = -h + \left\{ L(t)\mathcal{H}(t(\tau - t)) + l_{\max}\mathcal{H}(t - \tau) \right\} H(r, \theta), \tag{1}$$

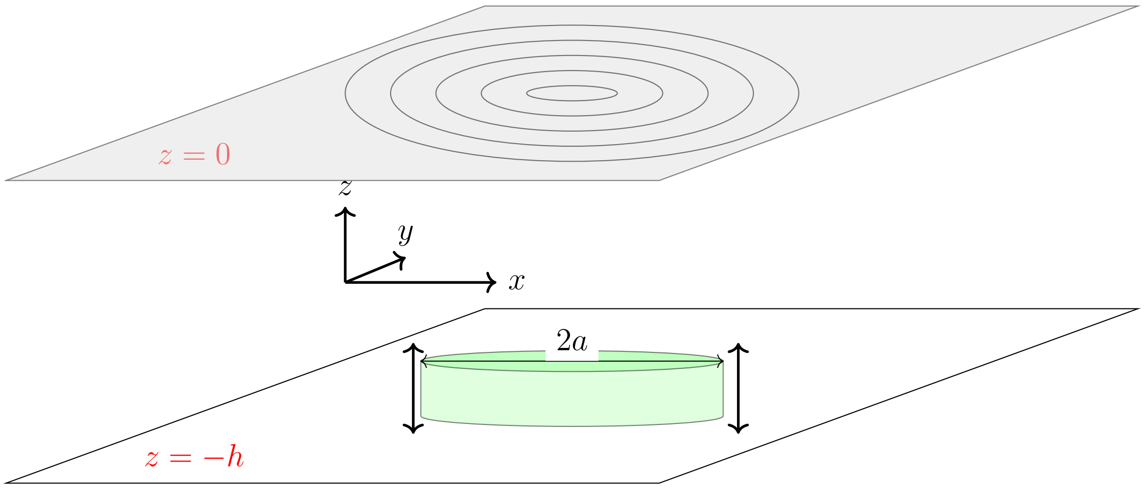

Figure 1: Schematic diagram of the physical problem in a compressible ocean having a flat circular rising bottom.

where

$$H(r,\theta) = \begin{cases} 1 & \text{if} \quad r < a, \\ 0 & \text{if} \quad r > a \end{cases} \tag{2}$$

In addition, $\dfrac{dL}{dt} = l(t)$ with $L(0) = 0$ and $L(\tau) = l_{\max}$. Now defining a velocity potential $\Phi(r,\theta,z,t)$ in the fluid region, the boundary value problem (BVP) can be written as

$$\nabla^2_{(r,\theta,z)}\Phi = \frac{1}{c^2}\left(\Phi_{tt} + g\Phi_z\right) \qquad \text{in} \quad -h < z < 0, \tag{3a}$$

$$\Phi_{tt} + g\Phi_z = 0 \qquad \text{at} \quad z = 0, \tag{3b}$$

$$\Phi_z = l(t)H(r,\theta)\mathcal{H}(t(\tau - t)) \qquad \text{at} \quad z = -h, \tag{3c}$$

where $\nabla^2_{(r,\theta,z)}$ represents the Laplacian operator in (r,θ,z) space.

1.2 Solution

Applying Fourier transformation in the time variable of the form

$$\phi(r,\theta,z,\omega) = \int_0^\infty \Phi(r,\theta,z,t)e^{-\mathrm{i}\omega t}dt, \tag{4a}$$

whose inverse transform is given by

$$\Phi(r,\theta,z,t) = \frac{1}{2\pi}\int_0^\infty \phi(r,\theta,z,\omega)e^{\mathrm{i}\omega t}d\omega, \tag{4b}$$

The boundary value problem (BVP) will be converted to

$$\nabla^2_{(r,\theta,z)}\phi = \frac{g}{c^2}\phi_z - \frac{\omega^2}{c^2}\phi \qquad \text{in} \quad -h < z < 0, \tag{5a}$$

$$-\omega^2\phi + g\phi_z = 0 \qquad \text{at} \quad z = 0, \tag{5b}$$

$$\phi_z = \begin{cases} \zeta_0(\omega) & \text{if} \quad r < a, \\ 0 & \text{if} \quad r > a \end{cases} \qquad \text{at} \quad z = -h, \tag{5c}$$

where

$$\zeta_0(\omega) = \int_0^\infty l(t) e^{-\mathrm{i}\omega t} dt$$

The above BVP is equivalent to the BVP corresponding to the motion of a portion of the ocean bottom oscillating vertically with a maximum amplitude of $\zeta_0(\omega)$. There are two solutions corresponding to $r > a$ and $r < a$. These two solutions differ by a particular solution f_p, a function of z only, due to non-zero boundary condition (5c) for $r < a$. Let us first find $f_p(z)$ with $\zeta_0(\omega) = 1$. The BVP to find the particular solution turns out to be

$$f_p'' - \gamma \phi_p' + \frac{\omega^2}{c^2} \phi_p = 0, \tag{6a}$$

$$f_p' = \frac{\omega^2}{g} \phi_p, \quad z = 0, \tag{6b}$$

$$f_p' = 1, \quad z = -h \tag{6c}$$

The final form of f_p is

$$\phi_p(z) = \zeta_0(\omega) \frac{e^{\gamma(z+h)/2} \left[\left(\frac{\omega^2}{g} - \frac{\gamma}{2} \right) \sinh(k_s z) + k_s \cosh(k_s z) \right]}{\left(\frac{\omega^2}{c^2} - \frac{\omega^2 \gamma}{2g} \right) \sinh(k_s h) + \frac{\omega^2 k_s}{g} \cosh(k_s h)}, \quad \text{where } k_s^2 = (\gamma/2)^2 - \omega^2/c^2.$$

The solution of BVP with the homogeneous boundary condition at $z = -h$ is given by (assuming symmetry in the θ coordinate)

$$\phi(r, z, \omega) = \sum_{n=0}^{\infty} \alpha_n K_0(k_n r) f_n(z). \tag{7}$$

for some arbitrary constants α_n and f_ns are defined by

$$f_n(z) = \frac{e^{(\gamma z/2)} \left(\frac{\gamma}{2} \sin \mu_n(z+h) - \mu_n \cos \mu_n(z+h) \right)}{\gamma \sin(\mu_n h) - 2\mu_n \cos(\mu_n h)}, \tag{8}$$

where μ_n satisfies the dispersion relation

$$\frac{\omega^2}{g} = -\mu \frac{\left[1 + \left(\frac{\gamma}{2\mu} \right)^2 \right] \tan \mu h}{1 - \left(\frac{\gamma}{2\mu} \right) \tan \mu h}. \tag{9}$$

with $\mu^2 = (\gamma/2)^2 + k^2 - \omega^2/c^2$. Hence we write the potentials in the regions $r > a$ and $r < a$ as

$$\phi_{in}(r, z, \omega) = \sum_{n=0}^{\infty} A_n I_0(k_n r) f_n(z) + \zeta_0 f_p(z), \qquad \phi_{out}(r, z, \omega) = \sum_{n=0}^{\infty} B_n K_0(k_n r) f_n(z). \tag{10}$$

Applying the boundary conditions at $r = a$,

$$A_n I_0'(k_n a) - B_n K_0'(k_n a) = 0 \qquad \text{for } n = 0, 1, 2, \ldots. \tag{11a}$$

$$\text{and} \quad A_n I_0(k_n a) - B_n K_0(k_n a) = -\zeta_0 \frac{\langle f_n, f_p \rangle_c}{\langle f_n, f_n \rangle_c}, \qquad \text{for } n = 0, 1, 2, \ldots, \tag{11b}$$

where the inner product $\langle f_n, f_p \rangle_c$ is defined by by

$$\langle f_n,\, f_p\rangle_c := \int_{-h}^{0} e^{-\gamma z} f_n(z) f_p(z)\, dz$$
$$= \left\{ \frac{\omega^2}{g} \left(\frac{\cosh \mu_n h - \cosh k_s h}{\mu_n^2 - k_s^2} \right) - \left[\frac{\gamma}{2\mu_n} \left(\frac{\omega^2}{g} - \frac{\gamma}{2} \right) + \mu_n \right] \frac{\sinh \mu_n h}{\mu_n^2 - k_s^2} \right.$$
$$\left. + \left[\frac{\gamma}{2k_s} \left(\frac{\omega^2}{g} - \frac{\gamma}{2} \right) + k_s \right] \frac{\sinh k_s h}{\mu_n^2 - k_s^2} \right\} \Big/ \mathcal{F}_n(\omega, \gamma) \tag{12a}$$

where

$$\mathcal{F}_n(\omega,\gamma) = \left\{ \left(\frac{\omega^2}{c^2 k_s} - \frac{\omega^2 \gamma}{2 g k_s} \right) \sinh\left(k_s h\right) + \frac{\omega^2}{g} \cosh\left(k_s h\right) \right\} \left(\frac{\gamma}{2\mu_n} \sinh\left(\mu_n h\right) - \cosh\left(\mu_n h\right) \right) e^{\frac{-\gamma h}{2}}.$$

Similarly, the other inner product is defined by and equals

$$\langle f_n,\, f_m\rangle_c := \int_{-h}^{0} e^{-\gamma z} f_n(z) f_m(z)\, dz,$$
$$= \frac{\frac{\sinh 2\mu_n h}{4\mu_n} \left(\frac{\gamma^2}{4\mu_n^2} + 1 \right) + \frac{h}{2} \left(1 - \frac{\gamma^2}{4\mu_n^2} \right) - \frac{\gamma}{4\mu_n^2} \left(\cosh 2\mu_n h - 1 \right)}{\left(\frac{\gamma}{2\mu_n} \sinh\left(\mu_n h\right) - \cosh\left(\mu_n h\right) \right)^2} \delta_{mn}. \tag{13}$$

Subsequently, the unknown coefficients A_n and B_n are written as

$$A_n = -a\zeta_0 k_n \frac{\langle f_n, f_p\rangle_c}{\langle f_n, f_n\rangle_c} K_1(k_n a) \quad \text{and} \quad B_n = a\zeta_0 k_n \frac{\langle f_n, f_p\rangle_c}{\langle f_n, f_n\rangle_c} I_1(k_n a) \tag{14a}$$

Finally, taking the inverse Fourier transformation defined in Eq. 4b, we get back the time-dependent potential function and subsequently derive the surface elevation with z-derivative. The numerical results will be shown during the presentation.

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