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Water wave Green function with compressibility effect

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The fundamental solution of potential flows in water of finite depth including the full effect of fluid compressibility is formulated by applying the Hankel transform and analysed by developing its eigenvalue series based on the partial fraction expansion of the integrand function. Special attention is paid on the limit of low wave frequency where the classical solution ignoring fluid compressibility presents an important issue. The linear potential at the limit for an oscillating frequency approaching to zero is not equal to that at zero frequency. This inconsistency yields unphysical results that the added-mass coefficients tend to infinity and radiation damping to a non-zero value at zero oscillating frequency. It is shown that the new Green function is continuous in the full range of low frequencies and consistent at the limit of zero frequency. Application of this new Green function in radiation-diffraction codes is expected to generate correct coefficients of added-mass and damping at low frequencies.

1 Introduction

Controversy results for the added-mass coefficient a_{33} and the damping b_{33} of a half-immersed circular cylinder (2D) in water of finite depth have been obtained in [1] by using the method of eigenvalue expansions. Both the value of $a_{33} < \infty$ and $b_{33} > 0$ tend to a finite value at the limit of low frequency, unlike in deepwater case, $a_{33} \to \infty$ and $b_{33} \to 0$. The same 2D case is analysed in [2] to figure out a constant exists and depends on the cylinder radius relative to the waterdepth between a_{33} for a halved cylinder at the mean free surface and that for the same halved cylinder inversely placed on the sea bottom, at the same limit of low frequency.

In water of finite depth, the 3D case of a hemisphere studied in [3] shows that a_{33} tends to infinity and b_{33} to a positive finite value at the low frequency. These results are manifestly non-physical. Indeed, computed at the very zero frequency, the added-mass coefficient a_{33} is of finite value and the damping coefficient b_{33} is nil. In fact, the exigence of a finite value for all added-mass coefficients and zero for all damping coefficients at the zero frequency is physically accepted in all cases including 2D and 3D in water of either infinite or finite depth.

Further analysis in [3] shows that the fundamental solution (Green function) used in boundary element methods has exactly the same behaviour that the real part of the Green function tends to infinity while the imaginary part to a non-zero finite value, instead to a purely real value at the zero frequency. Thus in the present study, we start with the study on the fundamental solution of water waves associated with a pulsating source by including the fluid compressibility. A new Green function is then formulated and analysed. Some discussions are followed afterwards.

2 Green function with fluid compressibility

We define a Cartesian coordinate system (x, y, z) with the horizontal plane (x, y) at the mean free surface $(z = 0)$ and the z-axis oriented upwards so that the sea bottom is represented by $z = -1$ (all coordinates are scaled with the waterdepth h). The oscillating frequency, the acceleration due to gravity and the sound speed are denoted by ω, g and c, respectively. In relation with these basic parameters, we have $\nu = \omega^2 h/g$ and two important parameters associated with fluid compressibility $\Omega = \omega h/c$ and $\gamma = gh/(2c^2)$. We consider the time-harmonic potential $G(P, Q)$ at the field point $P(x, y, z)$ generated by a pulsating source locate at the point $Q(x', y', z')$. The Green function $G(P, Q)$ should be a function of (r, z, z') with the horizontal distance $r = \sqrt{(x - x')^2 + (y - y')^2}$. The set of equations including the governing equation in the domain and boundary conditions on the free surface and on the sea bottom, can be found in [4], and reported here

$$\begin{aligned} (\nabla^2 + \Omega^2 - 2\gamma\partial_z)G(r, z, z') &= -4\pi\delta(P - Q) && z < 0\,; \quad z' < 0 \\ \partial_z G(r, z, z') &= 0 && z = -1 \\ (-\nu + \partial_z)G(r, z, z') &= 0 && z = 0 \end{aligned} \tag{1}$$

and a radiation condition $G(r, z, z') \to 0$ for $r \to \infty$, to close the problem. The free-space solution to the governing equation (first line) in (1) without considering the boundary conditions (second and third lines) in (1) is written as

$$G_0(r, z, z') = \mathrm{e}^{\gamma(z-z')}\frac{\mathrm{e}^{-\mathrm{i}\alpha R}}{R} = \mathrm{e}^{\gamma(z-z')}\int_0^\infty \frac{\mathrm{e}^{-\mu|z-z'|}}{\mu} J_0(kr)k\,\mathrm{d}k \tag{2}$$

with $R = \sqrt{r^2 + (z - z')^2}$. From above (2), the parameter α can be considered as the wavenumber in the spheric radial direction associated with the compressibility while k is the wavenumber in the horizontal direction and μ characterizes the variation in the vertical direction. These three wavenumbers satisfy the classical relationship

$$k^2 = \mu^2 + \alpha^2 \quad \text{with} \quad \alpha^2 = \Omega^2 - \gamma^2 = \gamma(2\nu - \gamma) \tag{3}$$

determined by the governing equation. The integral representation on the right hand side of (2) is derived by the Hankel transform. To satisfy the boundary condition on the sea bottom, we construct the so-called Rankine part

$$G^R(r, z, z') = \mathrm{e}^{\gamma(z-z')} \int_0^\infty \left[\left(\frac{\mu - \gamma}{\mu} \right) \frac{\mathrm{e}^{-\mu|z-z'|}}{\mu} + \left(\frac{\mu + \gamma}{\mu} \right) \frac{\mathrm{e}^{-\mu(z+z'+2)}}{\mu} \right] J_0(kr) k \, \mathrm{d}k \tag{4}$$

To satisfy the boundary condition on the free surface (third line) in (1), an additional part $G^F(r, z, z')$ is assumed to be represented by

$$G^F(r, z, z') = \mathrm{e}^{\gamma(z-z')} \int_0^\infty \frac{\mathcal{F}(\mu, z, z')}{\mu} J_0(kr) k \, \mathrm{d}k \tag{5}$$

By imposing $\partial_z G^F = 0$ at $z = -1$, and $(-\nu + \partial_z) G^F = -(-\nu + \partial_z) G^R$ on $z = 0$, the integrand function is derived

$$\mathcal{F}(\mu, z, z') = 2 \left(\frac{\nu + \mu - \gamma}{\mu} \right) \frac{\mathrm{e}^{-\mu} Z(z+1) Z(z'+1)}{D(\mu)} \tag{6}$$

with

$$\begin{aligned} Z(z+1) &= \mu \cosh \mu(z+1) - \gamma \sinh \mu(z+1) \\ Z(z'+1) &= \mu \cosh \mu(z'+1) - \gamma \sinh \mu(z'+1) \end{aligned} \tag{7}$$

and the dispersion function

$$D(\mu) = (\mu^2 - \gamma^2) \sinh \mu - \nu(\mu \cosh \mu - \gamma \sinh \mu) \tag{8}$$

Putting the Rankine part (4) and the free-surface part (5) together, we have

$$G(r, z, z') = G^R(r, z, z') + G^F(r, z, z') = \mathrm{e}^{\gamma(z-z')} \fint_0^\infty \frac{\mathcal{G}(\mu, z, z')}{\mu} J_0(kr) k \, \mathrm{d}k \tag{9}$$

with the integrand function

$$\mathcal{G}(\mu, z, z') = 2 \left(\frac{\mu - \gamma}{\mu} \right) \begin{cases} \dfrac{\mu \cosh \mu z + (\nu - \gamma) \sinh \mu z}{D(\mu)} Z(z'+1) & z > z' \\[2ex] \dfrac{\mu \cosh \mu z' + (\nu - \gamma) \sinh \mu z'}{D(\mu)} Z(z+1) & z < z' \end{cases} \tag{10}$$

which has poles determined by the dispersion equation $D(\mu) = 0$ with $D(\mu)$ defined by (8). Indeed, there are a real one $\mu = \mu_0$ and an infinity number of imaginary ones $\mu = \mathrm{i}\mu_n$ called evanescent modes. For $\nu \ll 1$, we have

$$\begin{aligned} \mu_0 &= \sqrt{\bar{\mu}_0^2 + \gamma^2 - \nu\gamma} + O(\nu^2) &&= \gamma + \nu(1-\gamma)/(2\gamma) + O(\nu^2) \\ \mu_n &= \sqrt{\bar{\mu}_n^2 + 2\nu\gamma^2/(n\pi)^2} + O(\nu^2) &&= n\pi - \nu/(n\pi)[1 - \gamma^2/(n\pi)^2] + O(\nu^2) \end{aligned} \quad \text{with} \quad \begin{aligned} \bar{\mu}_0 \tanh \bar{\mu}_0 &= \nu \\ -\bar{\mu}_n \tan \bar{\mu}_n &= \nu \end{aligned} \tag{11}$$

where $(\bar{\mu}_0, \bar{\mu}_n)$ are wavenumbers in the pure-gravity case. From above (11), it is remarkable that $\mu_0 = \gamma > 0$ unlike $\bar{\mu}_0 = \sqrt{\nu} \to 0$ and that the evanescent wavenumbers $\mu_n \approx \bar{\mu}_n$ for $n \geq 1$. Due to the presence of $\mu = \mu_0$, there is a pole at $k = k_0 = \sqrt{\mu_0^2 + \alpha^2}$, and the integral along the real k-axis should be deformed along the path circumventing below the pole represented by $\fint(\cdots)\,\mathrm{d}k$, in order to satisfy the radiation condition.

3 Series representation of the Green function

The integrand function $\mathcal{G}(\mu, z, z')$ defined by (10) has a finite value for $\mu \to 0$ and $\mathcal{G}(\mu, z < 0, z' < 0)$ tends to zero for $\mu \to \infty$ in the full complex μ plane, so that $\mathcal{G}(\mu, z, z')$, as a meromorphic function in the μ plane (not in k plane), can be expressed by the partial fraction expansion associated with all the poles in the complex μ plane, including the symmetric poles $\mu = -\mu_0$ and $\mu = -\mathrm{i}\mu_n$ with (μ_0, μ_n) for $n \geq 1$ determined by $D(\mu) = 0$, i.e.

$$\mathcal{G}(\mu, z, z') = \frac{c_0^+}{\mu - \mu_0} + \frac{c_0^-}{\mu + \mu_0} + \sum_{n=1}^{\infty} \left(\frac{c_n^+}{\mu - \mathrm{i}\mu_n} + \frac{c_n^-}{\mu + \mathrm{i}\mu_n} \right) \tag{12}$$

The coefficients $(c_0^\pm, c_n^\pm)$ are obtained by

$$c_0^\pm = \lim_{\mu\to\pm\mu_0} (\mu \mp \mu_0)\mathcal{G}(\mu, z, z') = \left(\frac{\mu_0 \mp \gamma}{\mu_0}\right) c_0 \; ; \quad c_n^\pm = \lim_{\mu\to\pm\mathrm{i}\mu_n} (\mu \mp \mathrm{i}\mu_n)\mathcal{G}(\mu, z, z') = \left(\frac{\mu_n \pm \mathrm{i}\gamma}{\mu_n}\right) c_n \tag{13}$$

with (c_0, c_n) given by

$$\begin{aligned} c_0 &= 2\mu_0 \frac{[\mu_0 \cosh\mu_0(z+1) - \gamma\sinh\mu_0(z+1)][\mu_0\cosh\mu_0(z'+1) - \gamma\sinh\mu_0(z'+1)]}{(\mu_0^2-\gamma^2)(2\mu_0+\sinh 2\mu_0) - 4\mu_0\gamma\sinh^2\mu_0 + 2\gamma^2\sinh 2\mu_0} \\ c_n &= 2\mu_n \frac{[\mu_n \cos\mu_n(z+1) - \gamma\sin\mu_n(z+1)][\mu_n\cos\mu_n(z'+1) - \gamma\sin\mu_n(z'+1)]}{(\mu_n^2+\gamma^2)(2\mu_n+\sin 2\mu_n) - 4\mu_n\gamma\sin^2\mu_n - 2\gamma^2\sinh 2\mu_n} \end{aligned} \tag{14}$$

From (11) for $\nu \ll 1$, we have $\mu_0 \approx \gamma$ and $c_0 \approx 1/2$ for $z = 0 = z'$. What is important concerns the coefficients $c_0^\pm$

$$c_0^+ \approx \nu c_0(1-\gamma)/(2\gamma^2) \; ; \quad c_0^- = c_0^+ + 2c_0\gamma/\mu_0 \approx c_0^+ + 1 \tag{15}$$

The partial fraction expansion (12) is then rearranged as

$$\mathcal{G}(\mu, z, z') = 2c_0^+ \left(\frac{\mu}{\mu^2-\mu_0^2}\right) + \gamma\frac{2c_0}{\mu_0}\left(\frac{1}{\mu+\mu_0}\right) + \sum_{n=1}^{\infty} 2c_n \left(\frac{\mu-\gamma}{\mu^2+\mu_n^2}\right) \tag{16}$$

which is introduced to (9) and the integrations in k space yield the series representation

$$G(r, z, z') = \mathrm{e}^{\gamma(z-z')} \left\{ \left[c_0^+ \mathrm{i}\pi H_0(k_0 r) + \sum_{n=1}^{\infty} 2c_n K_0(k_n r) \right] + \gamma \left[\frac{2c_0}{\mu_0}\tilde{H}(\mu_0 r) - \sum_{n=1}^{\infty} \frac{2c_n}{\mu_n}\tilde{K}(\mu_n r) \right] \right\} \tag{17}$$

in which $H_0(\cdot)$ is the zeroth order Hankel function of the first kind, and $K_0(\cdot)$ the zeroth order modified Bessel function of the second kind. The wavenumbers (k_0, k_n) in the first group of two terms are defined by

$$k_0 = \sqrt{\mu_0^2+\alpha^2} \approx \sqrt{\nu}\sqrt{1+\gamma} \; ; \quad k_n = \sqrt{\mu_n^2+\alpha^2} \approx n\pi - \nu/(n\pi)\big[1-\gamma(1-\gamma)-\gamma^2/(n\pi)^2\big] \tag{18}$$

for $\nu \to 0$. Furthermore, two additional functions $\tilde{H}(\mu_0 r)$ and $\tilde{K}(\mu_0 r)$ in (16) are defined and estimated here

$$\begin{aligned} \tilde{H}(\mu_0 r) &= \int_0^\infty \frac{J_0(kr)}{\mu(\mu+\mu_0)} k\,\mathrm{d}k < \frac{\pi}{2}\big[\mathbb{H}_0(\mu_0 r) - Y_0(\mu_0 r)\big] \\ \tilde{K}(\mu_n r) &= \int_0^\infty \frac{\mu_n J_0(kr)}{\mu(\mu^2+\mu_n^2)} k\,\mathrm{d}k < \frac{\pi}{2}\big[I_0(\mu_n r) - \mathbb{L}_0(\mu_n r)\big] \end{aligned} \tag{19}$$

by approximating μ by k since $\mu > k$ for $\nu < \gamma/2$. The integrals are evaluated by using (6.562.2) and (6.566.5) in [5]. The special functions in (19) are $\mathbb{H}_0(\cdot)$ the zeroth-order Struve function, $Y_0(\cdot)$ the zeroth-order Bessel of the second kind, $I_0(\cdot)$ the zeroth-order modified Bessel function of the first kind, and $\mathbb{L}_0(\cdot)$ the zeroth-order modified Struve function. All these special functions are explicitly defined in [6].

4 Discussion and conclusions

The series representation (17) contains four terms which are classified into two groups. The first group with two first terms looks like that in John's formulation obtained in [7]. The striking difference is the coefficient c_0^+ associated with $H_0(k_0 r)$ given in (18) which tends to zero for $\nu \to 0$ unlike that $c_0^+ \approx c_0 = 1/2$ if $\gamma = 0$ for $z = 0 = z'$. On the left part of Figure 1, the coefficient c_0^+ defined in (13) and c_0 is depicted against $\omega\sqrt{h/g} = \sqrt{\nu}$ in logarithmic scale for various waterdepth h =10m, 100m and 1000m. The dashed line represents c_0 and the dotted line for ν. The second group with two last terms is multiplied with the factor γ. The coefficient $\gamma 2c_0/\mu_0 \approx 1$ of the first term for $\nu \to 0$, is associated with the function $\tilde{H}(\mu_0 r)$ defined and estimated in (19), which is finite and of purely real since $\mu_0 \approx \gamma > 0$. The dominant terms in (17) can be represented by the function excluding the series of evanescent modes, written by

$$\tilde{G}(r, \gamma) = \left(\frac{\mu_0-\gamma}{\mu_0}\right) \mathrm{i}\pi H_0(k_0 r) + \frac{2\gamma}{\mu_0}\tilde{H}(\mu_0 r) \quad vs. \quad \tilde{G}(r, 0) = \mathrm{i}\pi H_0(k_0 r) \tag{20}$$

and depicted in the middle of Figure 1 for the real part and on the right for the imaginary part at a fixed $r = 1$. The real part of the function $\tilde{G}(r,\gamma)$ tends to a constant for $\nu \to 0$ while that of $\tilde{G}(r, \gamma = 0)$ goes to infinity. The imaginary part of $\tilde{G}(r,\gamma)$ for $\gamma > 0$ tends to zero while that of $\tilde{G}(r,\gamma = 0)$ tends to a non-zero constant.

In both groups on the right hand side of (17), the sums associated with c_n for $n \geq 1$ are real and finite for any value of $r > 0$. Indeed, due to the presence of $\gamma > 0$, the Green function $G(r, z, z')$ tends to a finite value which is

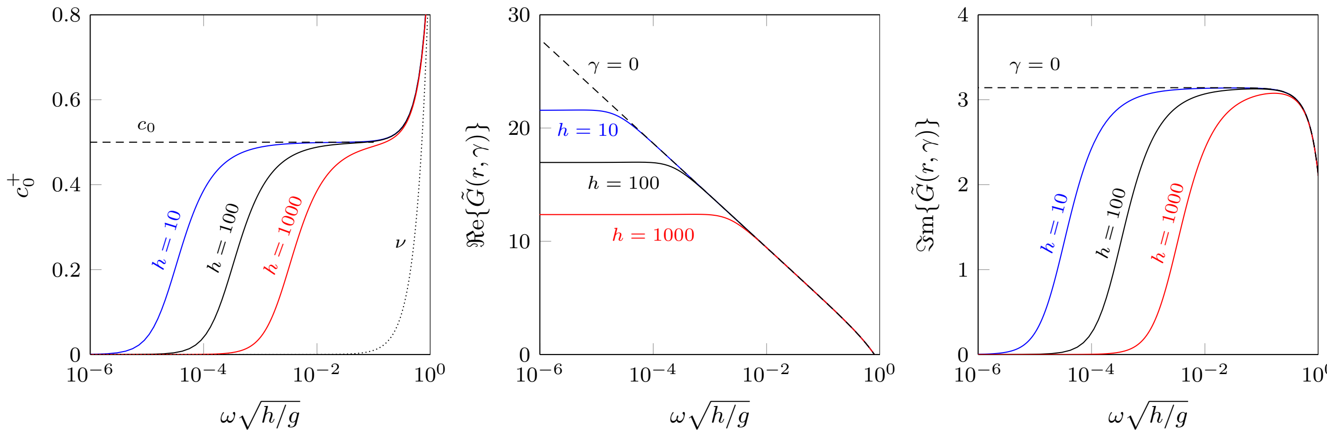

Figure 1: Coefficients of the partial fraction expansion c_0^+ defined by (13) versus c_0 defined by (14) for $z = 0 = z'$ on the left, the real part of the function $\tilde{G}(r = 1, \gamma)$ defined by (20) in the middle and the imaginary part of $\tilde{G}(r = 1, \gamma)$ on the right.

purely real, unlike the classical Green function examined in [3] which tends a complex value $\mathrm{i}\pi - 2\log k_0$ with an infinity real part for $k_0 \to 0$. In the general case of $\nu > \gamma/2$, the first group of terms in (17) is nearly identical to the classical John's formulation [7] without compressibility effect, and the second group of terms are negligible.

Comparing with the classical Green function in the incompressible water (pure-gravity), the Laplace operator (∇^2) in the governing equation (1) is augmented by ($\Omega^2 - 2\gamma\partial_z$). These additional operators prevent any constant other than zero in the domain even both the boundary conditions on $z = 0$ and on $z = -1$ are of Neumann type, i.e., $\partial_z G(r, z, z') = 0$, at the zero frequency. Since the small value of γ, the operator ($-\gamma\partial_z$) is ignored in almost all previous studies on acoustic-gravity waves. Without this operator, the solution should possess the same symptom that the limit at zero frequency tends to infinity for the real part and a finite value for the imaginary part.

Beside the variation of mass density due to oscillating accelerations characterized by the operator (Ω^2), the operator ($-2\gamma\partial_z$) represents physically the mass density variation in depth due to the gravity. Including this operator ($-2\gamma\partial_z$), all components of the Green function have the multiplicative factor $\mathrm{e}^{\gamma(z-z')}$ due to which the symmetrical property is lost when z and z' are interchanged, i.e. $G(r, z', z) \neq G(r, z, z')$. If the boundary element method is based on the direct application of the Green's theorems, this special property of the Green function with compressibility effect induces additional boundary integrals over all boundary surfaces with the integrand function defined as the product of the unknown velocity potential, the Green function and the parameter γ. These additional boundary integrals are negligible since the results are proportional to γ.

Assuming that the velocity potential around a body can be expressed by a source distribution on the hull which should be purely real for $\nu \to 0$, so that the ratio between the real and imaginary parts (phase) of the velocity potential is directly associated with the same ratio of the Green function. Indeed, the classical Green function tends to a logarithmic singularity for the real part and a non-zero constant for the imaginary part, as analysed in [3]. Correspondingly, the added-mass coefficients in the vertical modes (heave/roll/pitch) resultant from the integration of the real part of the potential around the body hull, tend to infinity, and the damping coefficients obtained by the hull integration of the imaginary part of the potential, tend to a non-zero constant. Now the new Green function with compressibility effect behaves correctly: the real part tending to a constant and imaginary part approaching to zero for $\nu \to 0$. A boundary element method involving this new Green function is expected to generate physically acceptable results at low frequencies, and in particular, finite values for the added-mass coefficients and zero for the damping coefficients at the limit of zero frequency.

References

[1] Ursell F. (1978) On the virtual-mass and damping coefficients for long waves in water of finite depth. *J. Fluid Mech.*, 78, part 1, 17-28.

[2] McIver P. & Linton C.M. (1991) The added mass of bodies heaving at low frequency in water of finite depth. *Applied Ocean Research*, 13 (1), 12-17.

[3] Xie C.M., Ouled Housseine Ch. & Chen X.B. (2022) A new formulation of the Green function in water of finite depth at low frequencies. *Applied Ocean Research*, 128, http://doi.org/10.1016/j.apor.2022.103357.

[4] Longuet-Higgins M.S. (1950) A theory of the origin of microseisms. *Phil. Trans. R. Soc. Lond.* A 243, 1-35.

[5] Gradshteyn I.S. & Ryzhik I.M. (2007) Tables of integrals, Series and Products. 7th Edition. *Academic Press.*

[6] Abramowitz M. & Stegun I.A. (1967) Handbook of mathematical functions. *Dover Publications.*

[7] John F. (1950) On the motion of floating bodies II: simple harmonic motions. *Comm. Pure Appl. Math.* 45-101.