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Energy balance relation for flexural-gravity wave scattering due to a crack in a floating ice sheet in a three-layer fluid in the context of blocking dynamics

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Highlights

- Investigation of flexural-gravity wave blocking dynamics in three-layer fluid in case of finite ocean depth.
- The dispersion relation reveals that wave blocking occurs either in the surface mode or in one of the internal modes.
- The energy balance relation is established for flexural-gravity wave scattering due to a crack in a floating ice sheet in the perspective of wave blocking.

1. Introduction

There is an upsurge in interest in the utilization of ocean space by developing very large floating structures to account for ocean space utilization. These floating structures can be used as an alternative to land reclamation. Besides, there is an interest in the development of multipurpose facilities in the context of an exclusive economic zone for promoting the Blue Economy ([1]). As a result, there is a growing interest in the hydroelastic analysis of very large floating structures for estimating the hydrodynamic performance of these structures under wave action. A parallel branch of study which is analogous to wave interaction with this kind of floating structure is the wave-ice interaction problems in the polar region. In both cases, the structure is modelled as a thin elastic plate. Surface gravity waves interact with these floating structures in generating flexural gravity waves. In extreme events such as high currents and wind, flexural gravity blocking occurs in the presence of high compressive force. There is a growing interest in recent years in the analysis of wave-structure interaction problems in the context of wave blocking (see [2]). The ocean exhibits density stratification due to changes in temperature and salinity along water depth. Typically, the sharp change in density leads to the generation of internal waves. These internal waves are noticed at various locations in the ocean, such as the Celtic Sea ([3]) and Andaman & Nicobar islands ([4]). Woodson [5] reviewed the impact of internal waves in nearshore ecosystems. In the present study, an attempt has been made to investigate flexural gravity wave blocking in a three-layer fluid having a plate-covered surface and two interfaces, assuming that the stratified ocean is modelled as a three-layer fluid of different densities.

2. Mathematical formulation

Flexural-gravity wave scattering due to a straight line crack in a floating ice sheet is analyzed in the presence of lateral compressive force in a three-layer fluid of finite depth. The physical problem is demonstrated under the assumption of linear water wave theory and the small amplitude structural response of the floating ice sheet. Here, two semi-infinite floating ice sheets are separated by a linear crack at $(0, 0)$ (see Figure 1) and the ice sheets are modelled as thin elastic plates. The physical problem is considered in the two-dimensional Cartesian coordinate system with $x-$axis being in the horizontal direction and the $y-$axis being in the vertical downward positive direction. The upper-layer fluid of constant density ρ_1 occupies the region $-\infty < x < \infty$, $0 < y < h_1$, with $y = 0$ being the mean ice-covered surface. The middle-layer fluid of constant density $\rho_2 (> \rho_1)$ occupies

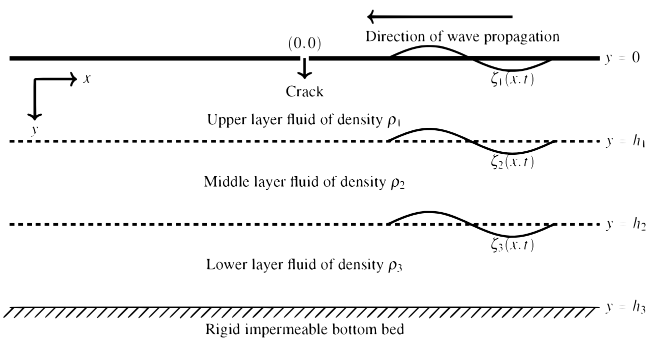

Figure 1: Floating ice sheet having a crack in a three-layer fluid.

the region $-\infty < x < \infty$, $h_1 < y < h_2$, with $y = h_2$ being the upper interface. The lower-layer fluid of density $\rho_3(> \rho_2 >, \rho_1)$ occupies the region $-\infty < x < \infty$, $h_2 < y < h_3$ with the lower interface being at $y = h_3$. The fluid is assumed to be inviscid and incompressible, and the flow is irrotational and simple harmonic in time with angular frequency ω which ensures the existence of a velocity potential $\Phi(x, y, t)$ of the form $\Phi(x, y, t) = \text{Re}\{\phi(x, y)e^{-\text{i}\omega t}\}$. Besides, the ice sheet deflection and interface elevations are of the forms $\zeta_j(x, t) = \text{Re}\left\{\eta_j(x)\text{e}^{-\text{i}\omega t}\right\}$ for $j = 1, 2, 3$. Thus, the spatial velocity potential $\phi(x, y)$ satisfies the partial differential equation

$$\nabla^2\phi = 0 \quad \text{in the fluid regions.} \tag{1}$$

Further, the linearized kinematic boundary conditions at the ice-covered surface and interfaces are of the form

$$\eta(x) \quad = \quad \frac{\text{i}}{\omega}\frac{\partial\phi(x, y)}{\partial y}, \tag{2}$$

with $\eta(x)$ being one of the $\eta_j(x)$ as appropriate. Besides, the boundary condition on the ice-covered surface is given by [6]

$$\left(D\frac{\partial^4}{\partial x^4} + Q\frac{\partial^2}{\partial x^2} + 1\right)\frac{\partial\phi(x, y)}{\partial y} + K\phi(x, y) = 0 \quad \text{on} \quad y = 0,\, x \in (-\infty, \infty) \setminus \{0\}, \tag{3}$$

where $D = EI/(\rho_1 g - d\rho_p\omega^2)$, $Q = N/(\rho_1 g - d\rho_p\omega^2)$, $K = \rho_1\omega^2/(\rho_1 g - d\rho_p\omega^2)$, $EI = Ed^3/12(1 - \nu^2)$, E is Young's modulus, d is plate thickness, ρ_p is plate density, ν is Poisson's ratio and N is the uniform compressive stress. The interface boundary conditions at the mean interfaces are given by

$$\frac{\partial\phi(x, y-)}{\partial y} \quad = \quad \frac{\partial\phi(x, y+)}{\partial y} \;\text{ on }\; y = h_j,\; \text{ for } j = 1, 2, \tag{4a}$$

$$s_j\left\{\frac{\partial\phi(x, y-)}{\partial y} + K\phi(x, y-)\right\} \quad = \quad \frac{\partial\phi(x, y+)}{\partial y} + K\phi(x, y+) \text{ on } y = h_j,\; \text{ for } j = 1, 2, \tag{4b}$$

with $s_j = \rho_j/\rho_{j+1}(< 1)$ for $j = 1, 2$. Finally, the impermeable rigid bottom boundary condition is given by

$$\frac{\partial\phi(x, y)}{\partial y} = 0 \;\text{ on } y = h_3. \tag{5}$$

Moreover, the continuity of velocity and pressure along the interface near the crack yield

$$\phi_x(0+, y) = \phi_x(0-, y) \;\text{ and }\; \phi(0+, y) = \phi(0-, y) \quad \text{for} \quad 0 < y < h_3. \tag{6}$$

Further, assuming the free-edge conditions (zero bending moment and shear stress) have complied near the crack, the edge conditions in terms of the velocity potential $\phi(x, y)$ satisfies

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi}{\partial y}\right) = 0 \quad \text{and} \quad D\frac{\partial^3}{\partial x^3}\left(\frac{\partial\phi}{\partial y}\right) + Q\frac{\partial^2\phi}{\partial x\partial y} = 0 \quad \text{as} \quad (x, y) \longrightarrow (0\pm, 0). \tag{7}$$

Finally, the far-field radiation condition is of the form

$$\phi(x, y) = \begin{cases} \displaystyle\sum_{n=1}^{5}\left(I_n\text{e}^{-\text{i}\epsilon_n k_n x} + R_n\text{e}^{\text{i}\epsilon_n k_n x}\right)F_n(y) & \text{as} \quad x \longrightarrow \infty, \\ \displaystyle\sum_{n=1}^{5} T_n\text{e}^{-\text{i}\epsilon_n k_n x}F_n(y) & \text{as} \quad x \longrightarrow -\infty, \end{cases} \tag{8}$$

where k_n for $n = 1, 2, 3$ are the propagating wave modes associated with the ice-covered surface within the blocking frequency under the assumption that blocking occurs on the ice-covered surface mode, whilst k_4 and k_5 correspond to the propagating wave modes in the upper and lower interfaces respectively. On the other hand, outside the blocking frequency, the propagating wave modes k_n for $n = 1, 4, 5$ are associated with the ice-covered surface, upper and lower interfaces respectively, whilst k_n for $n = 2, 3$ contribute to the non-propagating wave modes, and thus I_j, R_j and T_j are considered to be zero for $j = 2, 3$. Further, I_ns, R_ns and T_ns are the amplitudes of the incident, reflected and transmitted waves in the n^{th} mode respectively. Moreover, the incident wave amplitudes I_n are assumed to be known and chosen suitably taking into account the direction of wave propagation within the blocking frequencies as in [6], whereas the reflected and transmitted wave amplitudes are unknown complex constants and to be determined using boundary conditions along the vertical interfaces and the edge conditions near the crack. Besides, $\epsilon_n = \pm 1$ depends upon the group velocity $c_g(k_n) \gtrless 0$. Further, k_ns in k satisfy the dispersion relation as given by

$$\mathcal{G}(k) \equiv k - s_1 k - s_1 K \frac{(Dk^4 - Qk^2 + 1 - \gamma K)k - K \tanh kh}{(Dk^4 - Qk^2 + 1 - \gamma K)k \tanh kh - K} \\ - K \frac{s_2 K \tanh kh_{32} \tanh hh_{21} + \{K - (1 - s_1)k \tanh kh_{32}\}}{s_2 K \tanh kh_{32} + \{K - (1 - s_1)k \tanh kh_{32}\} \tanh kh_{21}} = 0, \quad (9)$$

with $h_{ij} = h_i - h_j$ for $i = j + 1,\ j = 1, 2$. Moreover, the vertical eigenfunctions $F_n(y)$s are of the forms

$$F_n(y) = \begin{cases} l_{1n}(y) & \text{for } 0 < y < h_1, \\ l_{2n}(y) & \text{for } h_1 < y < h_2, \\ s_2 K \cosh k_n(h_3 - y) & \text{for } h_2 < y < h_3, \end{cases} \quad (10)$$

with

$$l_{1n}(y) = -\frac{P(k_n)}{\mathcal{T}(k_n)} \left\{ (Dk_n^4 - Qk_n^2 + 1)k_n \cosh k_n y - K \sinh k_n y \right\},$$
$$l_{2n}(y) = s_2 K \sinh k_n h_{32} \sinh k_n(h_2 - y) + M(k_n) \cosh k_n(h_2 - y),$$
$$M(k_n) = K \cosh k_n h_{32} - (1 - s_2)k_n \sinh k_n h_{32}, P(k_n) = M(k_n) \sinh k_n h_{21} + s_2 K \sinh k_n h_{32} \cosh k_n h_{21},$$
$$\mathcal{T}(k_n) = (Dk_n^4 - Qk_n^2 + 1)k_n \cosh k_n h_1 - K \sinh k_n h_1.$$

3. Wave blocking in three-layer

The physical parameter values used for the numerical computations are Young's modulus $E = 5$ GPa, Poisson's ratio $\nu = 0.3$, ice density $\rho_p = 922.5$ kg m^{-3}, ice thickness $d = 1$ m, acceleration due to gravity $g = 9.81$ m sec^{-2}, density ratios are $s_1 = 0.85$, $s_2 = 0.80$ and water depths are $h_1 = 10$ m, $h_2 = 20$ m, $h_3 = 50$ m, which are the same as in [6]. In addition, inertia term involving $\gamma\omega^2 << 1$ is neglected as in [2]. The dispersion curves given in Eq. (9) are shown in figure 2 for various values of compressive force Q. The figure reveals the nature of the dispersion curve in the plane of ω-k, which is monotonically growing for the uncompressed ice. With an increase in compressive force, local extrema occur in the dispersion graph, which is associated with lower/higher wavenumber. The maxima and minima in wave frequency are referred to as primary and secondary blocking frequencies respectively, as in [2]. Further, the slope of the $k - \omega$ curve vanishes at the blocking frequency, which is equivalent to the cessation of energy propagation. Moreover, it has been noticed that wave blocking may occur in either of the modes for certain values of the frequency with fixed compressive force. Further, the dispersion curve can have a maximum of five distinct real roots within the limit of blocking frequencies, of which three exist either in the ice-covered surface, upper or lower interface modes, where wave blocking occurs. The blocking frequency shift to a lower frequency as compressive force increases.

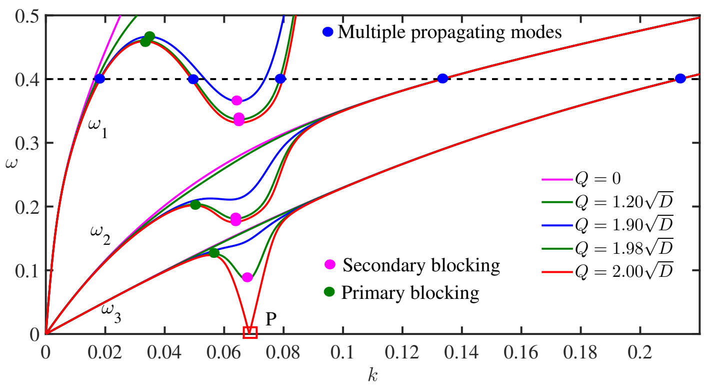

Figure 2: Dispersion graph demonstrates the occurrence of blocking for different values of lateral compressive force Q.

4. Energy balance relation

To establish the energy balance relation, using Green's integral theorem, as given by

$$\int_C \left(\phi^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial \phi^*}{\partial n} \right) ds = 0, \tag{11}$$

where C denotes the closed boundary of the three-layer fluid region represented as the union of two closed boundaries C_1 and C_2, ϕ^* is the complex conjugate of ϕ which satisfies Eqs. (1)-(7) and $\frac{\partial}{\partial n}$ represents the outward normal derivative to the closed boundary C. While the closed boundary C_1 consists of the horizontal ice-covered surface ($0 < x < X$; $y = 0$), the vertical boundary ($0 < y < \infty$; $x = X$), the bottom boundary ($0 < x < X$; $y = h_3$), and the vertical boundary at the crack ($0 < y < h_3$; $x = 0$), the closed boundary C_2 consists of the horizontal upper surface ($-X < x < 0$; $y = 0$), the vertical boundary ($0 < y < h_3$; $x = -X$), the bottom boundary ($-X < x < 0$; $y = h_3$), and the vertical boundary at the crack ($0 < y < h_3$; $x = 0$). Consequently, using Eqs. (3)-(8), the energy balance relation is obtained as

$$J_i |I_i|^2 \left\{ \left(K_{r1}^{(i)}\right)^2 + \left(K_{t1}^{(i)}\right)^2 - 1 \right\} + \sum_{m=4}^{5} J_m |I_m|^2 \left(K_{rm}^2 + K_{tm}^2 - 1\right) = 0 \quad \text{for} \quad i = 1, 2, \tag{12}$$

$$\begin{aligned}
\text{where } K_{r1}^{(i)} &= \sqrt{\sum_{j=1}^{3} \epsilon_j \mathcal{E}_{ij} \left| \frac{R_j}{I_i} \right|^2}, \ K_{t1}^{(i)} = \sqrt{\sum_{j=1}^{3} \epsilon_j \mathcal{E}_{ij} \left| \frac{T_j}{I_i} \right|^2} \ \text{for } i = 1, 2, \\
K_{rm} &= \left| \frac{R_m}{I_m} \right|, \ K_{tm} = \left| \frac{T_m}{I_m} \right| \ \text{for } m = 4, 5, \ \mathcal{E}_{ij} = \frac{J_j}{J_i} \ \text{for } j = 1, 2, 3, \\
J_n &= k_n \left[\left\{ \frac{P(k_n)}{\mathcal{T}(k_n)} K k_n \right\}^2 \frac{(2Dk_n^2 - Q)}{K} + L_n \right] \quad \text{for } n = 1, 2, ..., 5, \\
L_n &= s_1 s_2 \int_0^{h_1} F_n^2(y) dy + s_2 \int_{h_1}^{h_2} F_n^2(y) dy + \int_{h_2}^{h_3} F_n^2(y) dy \quad \text{for } n = 1, 2, ..., 5.
\end{aligned}$$

Here, $K_{r1}^{(i)}$ and $K_{t1}^{(i)}$ denote the generalized reflection and transmission coefficients with the incident wave being in the i^{th} mode with blocking occurring in the ice-covered surface mode. Besides, K_{rm} and K_{tm} are the reflection and transmission coefficients associated with the upper ($m = 4$) and lower ($m = 5$) interfaces, respectively. Similarly, the energy balance relation can be obtained when wave blocking occurs either at the upper or lower interfaces. Few results regarding the crack problem from the perspective of energy balance relation will be demonstrated during the presentation.

References

[1] P. Negi, S. Boral, and T. Sahoo. Scattering of long flexural gravity wave due to structural heterogeneity in the framework of wave blocking. *Wave Motion*, page 102949, 2022.

[2] S. C. Barman, S. Das, T. Sahoo, and M. H. Meylan. Scattering of flexural-gravity waves by a crack in a floating ice sheet due to mode conversion during blocking. *J. Fluid Mech.*, 916:A11, 2021.

[3] V. Vlasenko, N. Stashchuk, M. E. Inall, and J. E. Hopkins. Tidal energy conversion in a global hot spot: On the 3-d dynamics of baroclinic tides at the celtic sea shelf break. *J. Geophys. Res.: Oceans*, 119(6):3249–3265, 2014.

[4] A. R. Osborne and T. L. Burch. Internal solitons in the Andaman sea. *Science*, 208(4443):451–460, 1980.

[5] C. B. Woodson. The fate and impact of internal waves in nearshore ecosystems. *Annual review of marine science*, 10(1), 2018.

[6] S. C. Barman, S. Das, T. Sahoo, and M. H. Meylan. Scattering of flexural-gravity waves due to a crack in a floating ice sheet in a two-layer fluid in the context of blocking dynamics. *Phys. Fluids*, 34(5):056602, 2022.