# Filon Integration of Highly Oscillatory and Other Functions with Applications in Marine Hydrodynamics

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## HIGHLIGHTS

A generalized form of one-dimensional Filon quadrature is presented along with a few simple examples. One practical use case involves conversion between frequency- and time-domain radiation coefficients in ship seakeeping. A second example is Michell's wave resistance formula.

## 1 Filon Quadrature in One Dimension

Our primary focus is the numerical integration of highly oscillatory functions, though more generally an integral of the form

$$I = \int f(x)g(x)dx \tag{1}$$

where $f(x)$ is some discretized function and $g(x)$ is a well-defined, continuous function. The functions can be multivariate, though we first focus on one-dimensional integrals.

We assume the function $f(x)$ is piecewise linear over some range, i.e.,

$$f(x) = f_j + \frac{x - x_j}{h}(f_{j+1} - f_j) \qquad \text{for} \quad x_j \leq x \leq x_{j+1} \tag{2}$$

where $f_j = f(x_j)$ and $h = x_{j+1} - x_j$ is the step size. The integral over this range is

$$I_j = \int_{x_j}^{x_{j+1}} f(x)g(x)dx = f_{j+1}g'_{j+1} - f_j g'_j - \frac{f_{j+1} - f_j}{h}\left[g''_{j+1} - g''_j\right] \tag{3}$$

where shorthand $g'(x) = \int g(x)dx$ and $g''(x) = \int g'(x)dx$ has been used. The expression in Eqn. 3 can be used directly in a numerical integration scheme. The solution is exact if $f(x)$ is actually piecewise linear, though more generally we can apply this for discretized $f(x)$. There are practical issues, for example if the first or second integrals of $g(x)$ are singular or otherwise not well-defined.

### 1.1 Constant Integral $g(x) = 1$

We begin with a simple case with constant $g(x) = 1$. Substituting the above expression in Eqn. 3 gives the familiar trapezoidal rule

$$I_j = \frac{h}{2}(f_j + f_{j+1}) \tag{4}$$

Assuming integration limits of $0..Nh$ with constant spacing $h$ over the intervals gives total integral $I$ with coefficients reducing to a constant spacing trapezoidal scheme.

$$I = \sum_{j=0}^{N} I_j = h \sum_{j=0}^{N} C_j f(jh) \quad \text{with} \quad C_j = \begin{cases} \frac{1}{2} & \text{for} \quad j = 0, j = N \\ 1 & \text{for} \quad 0 < j < N \end{cases}$$

**1.2 Fourier Integral** $g(x) = e^{i\omega x}$
The work done on integrating highly oscillatory functions has direct application to Fourier-type integrals, going back to the work of Filon [1]. Using the continuous function $g(x) = e^{i\omega x}$ in Eqn. 3 and recalling that $e^{i\omega x_{j+1}} = e^{i\omega x_j} e^{i\omega h}$, we get

$$I_j = \frac{1}{\omega^2 h} \left[ f_j e^{i\omega x_j} \left(1 - e^{i\omega h} + i\omega h\right) + f_{j+1} e^{i\omega x_{j+1}} \left(1 - e^{-i\omega h} - i\omega h\right) \right] \tag{5}$$

If we assume the spacing $h$ is constant over all the intervals in the integral, the total integral over the range $-Nh..Nh$ can be written as

$$I = h \sum_{j=-N}^{N} C_j e^{i\omega jh} f(jh) \quad \text{with} \quad C_j = \begin{cases} \frac{1}{\omega^2 h^2}(1 - e^{i\omega h} + i\omega h) & \text{for} \quad j = -N \\ \frac{1}{\omega^2 h^2}(2 - e^{i\omega h} - e^{-i\omega h}) & \text{for} \quad j \neq \pm N \\ \frac{1}{\omega^2 h^2}(1 - e^{-i\omega h} - i\omega h) & \text{for} \quad j = +N \end{cases}$$

resulting in an weighting coefficients equivalent to those found in [2]. A series expansion shows that the weighting coefficients approach their trapezoidal equivalent as $\omega h \to 0$.

**1.3 Exponential Integral** $g(x) = e^{kx}$
An exponential function is another example that can numerically integrated using this approach. Though it is not oscillatory, some areas of high curvature can lead to errors when using simple trapezoidal integration schemes. Substituting $g(x) = e^{kx}$ into Eqn. 3 and using $e^{kx_{j+1}} = e^{kx_j} e^{kh}$ where $h = x_{j+1} - x_j$ gives

$$I_j = \frac{1}{k^2 h} \left[ f_j e^{kx_j} \left(-kh - 1 + e^{kh}\right) + f_{j+1} e^{kx_{j+1}} \left(kh - 1 + e^{-kh}\right) \right] \tag{6}$$

Following the same approach as before, assuming the spacing $h$ is constant over the integration limits $0..Nh$ yields the total integral

$$I = h \sum_{j=0}^{N} C_j e^{kjh} f(jh) \quad \text{with} \quad C_j = \begin{cases} \frac{1}{k^2 h^2}(e^{kh} - kh - 1) & \text{for} \quad j = 0 \\ \frac{1}{k^2 h^2}(e^{kh} + e^{-kh} - 2) & \text{for} \quad 0 < j < N \\ \frac{1}{k^2 h^2}(e^{-kh} + kh - 1) & \text{for} \quad j = N \end{cases}$$

which is equivalent to the form given in [3]. Again, the coefficients tend to trapezoidal terms in the limit of $kh \to 0$.

**2 Impulse Response Functions for Wave Radiation**
Classical ship seakeeping involves a linear system of equations in the frequency domain. Converting to time-domain impulse response functions (IRFs) is well known in the widely cited work of Cummins [4]. A form of the Cummins equation is shown below, using the frequency-domain damping $B(\omega)$ as the source data, which is a good candidate for Filon-trapezoidal integration.

$$K(t) = \frac{2}{\pi} \int_0^\infty (B(\omega) - B^\infty) cos(\omega t) d\omega \tag{7}$$

$$f(\omega) = B(\omega) - B^\infty \qquad g(\omega) = cos(\omega t) \tag{8}$$

Consider an example where the kernel and corresponding damping are analytically defined

$$K(t) = e^{-0.2t}\cos(0.6t) \quad \text{and} \quad B(\omega) = \frac{2 + 5\omega^2}{(2 - 6\omega + 5\omega^2)(2 + 6\omega + 5\omega^2)} \tag{9}$$

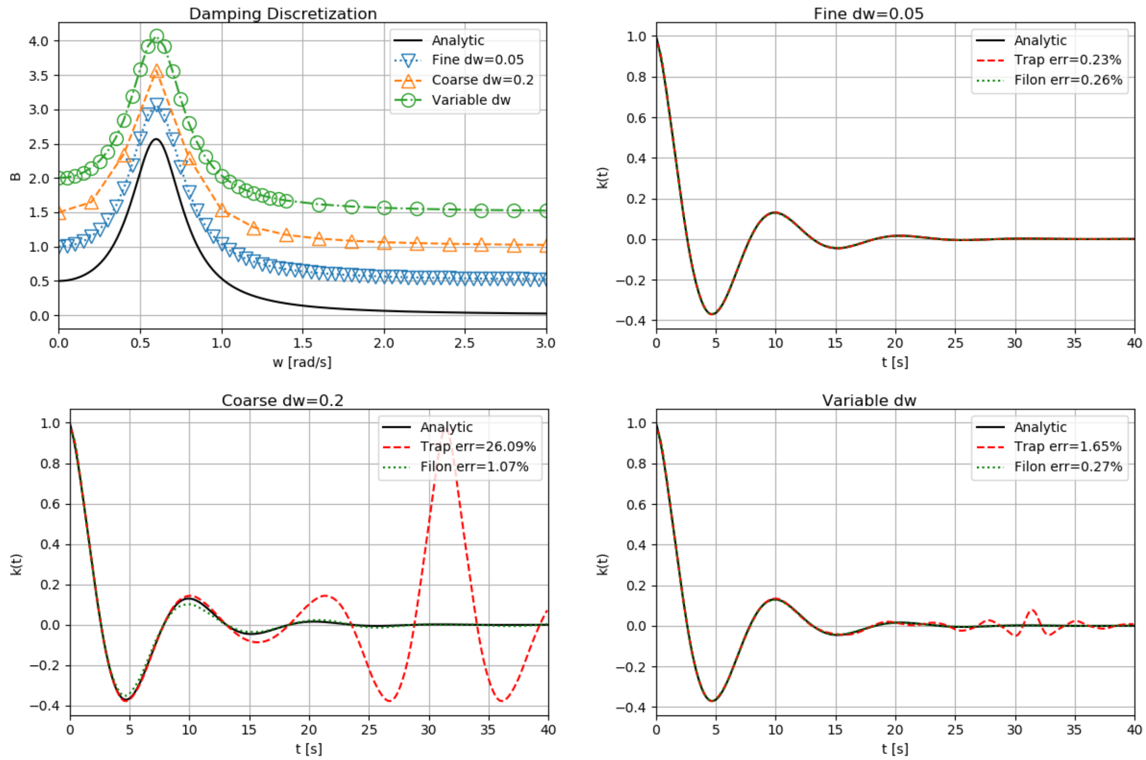


Figure 1: IRF computation with different frequency discretizations

The upper left of Figure 1 shows three different discretization schemes for the damping curve. Each set is used to numerically compute the IRF, as shown in the remaining sub-figures. The fine resolution shows very good accuracy for both trapezoidal and Filon integration, while using a coarse spacing gives very poor results with the trapezoidal approach. The variable resolution, i.e., finely resolved peak and otherwise coarse, also shows the Filon approach to have better accuracy.

The analytic case given here confirms that the Filon-trapezoidal integration is an improvement over the simple trapezoidal approach when numerically computing IRFs from wave radiation coefficients. The ability to accurately handle variable spacing should not be under-appreciated - many modern seakeeping panel codes take significant computational effort to achieve good quality results, so using a suitable integration scheme for time/frequency domain conversion is valuable. For example, we may want to focus computational effort to resolve peak responses and use fewer points to resolve the high-frequency tail.

## 3 Michell's Wave Resistance

One useful application of this integration scheme (and source of inspiration for this manuscript) is the evaluation of Michell's wave resistance formula [5]. The derivation of the integral can be found in Chapter 5 of [6]. One of the expressions that must be evaluated is the complex-valued free-wave spectrum $A(\theta)$, given by

$$A(\theta) = -i\frac{2}{\pi}k_0^2 \sec^4\theta \iint Y(x,z)e^{k_0 z \sec^2\theta}e^{ik_0 x \sec\theta}dxdz \tag{10}$$

where $k_0 = g/U^2$ and the double integral is evaluated on the center-plane.

The wave elevation at any point in the downstream far field is given by

$$\zeta(x,y) = \Re\int_{-\pi/2}^{\pi/2} A(\theta)\exp^{-ik(\theta)(x\cos\theta+y\sin\theta)}\,d\theta \tag{11}$$

and the total energy in the steady wave pattern is

$$R = \frac{\pi}{2}\rho U^2 \int_{-\pi/2}^{\pi/2} |A(\theta)|^2 \cos^3\theta d\theta \tag{12}$$

Following the conventions in [3] and [6], we define the double integral terms of the free-wave spectrum by

$$A(\theta) = -i\frac{2}{\pi}k_0^2 \sec^4\theta(P(\theta)+iQ(\theta)) \tag{13}$$

$$P(\theta)+iQ(\theta) = \int F(x,\theta)e^{ik_0 x\sec\theta}dx \quad \text{with} \quad F(x,\theta) = \int Y(x,z)e^{k_0 z\sec^2\theta}dz \tag{14}$$

where $k(\theta) = k_0\sec^2\theta$. The center-plane integral in Eqn. 10 has been replaced by two line integrals in Eqn. 14, suitable for integration using the one-dimensional Filon quadrature schemes presented earlier. The wave elevation and wave resistance can be numerically integrated using the one-dimensional Filon quadrature formulation.

## REFERENCES


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