Revisiting the classical deep-water wave instabilities – a discrete Hamiltonian approach

David Andrade, Raphael Stuhlmeier
Centre for Mathematical Sciences, University of Plymouth, Plymouth, United Kingdom
raphael.stuhlmeier@plymouth.ac.uk

HIGHLIGHTS Our aim is to provide a new perspective on the classical instabilities of surface water waves in deep water: the Type Ia and Ib instabilities arising from four-wave interactions (including the Benjamin-Feir instability (BFI)). We show that these can each be recast as a planar Hamiltonian dynamical system, and their dynamics understood without linearisation from the framework of the discrete Zakharov equation. Instability is characterised by the appearance of fixed-points in the phase plane, and certain heteroclinic orbits are identified as new discrete breather solutions.

1 Introduction

The stability of waves in deep water has traditionally been approached via linearisation, starting with various model equations, such as the nonlinear Schrödinger equation or (without restriction to narrow bandwidth) the Zakharov equation [1]. The simplest cases involve the interaction of four waves which can be captured using the Hamiltonian description of Krasitskii [2].

In the usual mathematical sense, instability requires that we start from a solution to a set of equations, and describes the evolution of perturbations to the solution. A handful of explicit solutions – the monochromatic (Stokes') wave and bichromatic wave train – form the backbone of classical instability results. The former is associated with the BFI and Type Ib instability, the latter with Type Ia, following the classification of Leblanc [3].

2 Discrete Hamiltonian formulation

We begin with the reduced Hamiltonian of the water wave problem in the discrete formulation, and up to third order in nonlinearity, cf. [2, Eq. (2.22)]:

$$H(b_1 \dots b_N, b_1^* \dots b_N^*) = \sum_{i=1}^{N} \omega_i |b_i|^2 + \frac{1}{2} \sum_{i,j,m,l=1}^{N} T_{ijkl} \delta_{ij}^{kl} b_i^* b_j^* b_k b_l. \tag{1}$$

Subscripts denote the dependence on wavenumber, i.e. $b(\mathbf{k}_j, t) = b_j$, * is the complex conjugate. The linear dispersion relation for deep water waves is $\omega_i = \sqrt{g\|\mathbf{k}_i\|}$, with g the gravitational acceleration. The Kronecker delta function is written $\delta_{ij}^{kl} = \delta(\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_k - \mathbf{k}_l)$, while the expression for the interaction kernel $T_{ijkl} = T(\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k, \mathbf{k}_l)$ can be found [2, 4]. We will use $\Gamma_i = T_{ii}|b_i|^2 + 2\sum_{j\neq i} T_{ij}|b_j|^2$ as a shorthand for the Stokes' correction terms (see [5]). The discrete Zakharov equation is obtained from this Hamiltonian via $i\frac{db_i}{dt} = \frac{\partial H}{\partial b_i^*}$.

Rewriting the Hamiltonian in amplitude and phase variables $b_i = |b_i| e^{i\phi_i}$,for $i = 1, \dots, N$, with a functional dependence on $|b_i|^2$ yields

$$\begin{aligned} H(|b_i|^2, \phi_i) = \sum_{i=1}^{N} \omega_i |b_i|^2 + \frac{1}{2} \sum_{i,j=1}^{N} e_{ij} T_{ij} |b_i|^2 |b_j|^2 + \\ + \frac{1}{2} \sum_{i,j=1}^{N} \sum_{k\neq i,j} \sum_{l\neq i,j} T_{ijkl} \sqrt{|b_i|^2 |b_j|^2 |b_k|^2 |b_l|^2} \delta_{ij}^{kl} \cos(\phi_i + \phi_j - \phi_k - \phi_l), \end{aligned} \tag{2}$$

where $e_{ij} = 1$ if $i = j$ and 2 otherwise. The first two terms on the right-hand side are trivial interactions responsible for Stokes'-type frequency corrections only (see [5]); the last term is responsible for energy exchanges among four modes. Moreover, it is clear that the phases occur only in a single *dynamic phase* combination $\theta_{ij}^{kl} = \phi_i + \phi_j - \phi_k - \phi_l$ for each discrete resonant set. An analogous decomposition of the Hamiltonian holds when quintet interaction terms are included, in which case the starting point is [2, Eq. (2.24)].

2.1 Reduction to a planar system

In addition to the Hamiltonian, the momentum $\mathbf{M} = \sum_{i=1}^{4} \mathbf{k}_i |b_i|^2$ and wave action $A = \sum_{i=1}^{4} |b_i|^2$ are conserved (see [2, Eq. (3.36)ff]). Together these conserved quantities can be used to cast each of the classical deep water instabilities in terms of a planar, Hamiltonian dynamical system.

Type Ib & Benjamin-Feir instability Type Ib instability is characterised by the resonance condition $2\mathbf{k}_1 = \mathbf{k}_3 + \mathbf{k}_4$, accompanied by a non-resonant satellite $\mathbf{k}_2$. The Benjamin-Feir instability is recovered when the amplitude of mode $\mathbf{k}_2$ is set to zero. The conservation laws and resonance condition are equivalent to the linear system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & 2k_1 - k_3 \\ l_1 & l_2 & l_3 & 2l_1 - l_3 \end{bmatrix} \begin{bmatrix} |b_1|^2 \\ |b_2|^2 \\ |b_3|^2 \\ |b_4|^2 \end{bmatrix} = \begin{bmatrix} A \\ M^x \\ M^y \end{bmatrix} \tag{3}$$

for $\mathbf{M} = (M^x, M^y)$, $\mathbf{k}_i = (k_i, l_i)$. The setting where only the carrier $\mathbf{k}_1$ and satellite $\mathbf{k}_2$ are present corresponds to a bichromatic wave solution to the Zakharov equation [4, Ch. 14.6] – or, if the satellite amplitude $|b_2(0)| = 0$ to the Stokes' wave solution [4, Ch. 14.5]. Either case fixes the momentum and total wave action, and thus a particular solution to (3). The general solution is then obtained by adding $f(t)(2, 0, -1, -1)$, an element in the kernel of the coefficient matrix. This is equivalent to introducing an amplitude parameter η such that $|b_1|^2 = A\eta$, $|b_2|^2 = A\eta$, $|b_3|^2 = A(1/2 - \eta)$, $|b_4|^2 = A(1/2 - \eta)$. Here $D = (|b_1|^2 + |b_3|^2 + |b_4|^2)/A$ is the fraction of wave action in the resonant triad and $B = |b_2|^2/A$ that of the satellite wave $\mathbf{k}_2$. Conservation of wave action means $B + D = 1$, and $\eta \in [0, D]$. In terms of η and $\theta = 2\phi_1 - \phi_3 - \phi_4$ we have a planar system

$$\frac{d\eta}{dt} = 2\eta T_{1134} A \,(D - \eta) \sin(\theta)\,, \tag{4}$$

$$\frac{d\theta}{dt} = 2AT_{1134} \cos(\theta)\,(D - 2\eta) + (\Omega_1 \eta + \Omega_0)\,A + \Delta_{11}^{34}, \tag{5}$$

with $2\Gamma_1 - \Gamma_3 - \Gamma_4 = A\Omega_0 + A\Omega_1\eta$ and $\Delta_{11}^{34} = 2\omega_1 - \omega_3 - \omega_4$

Type Ia instability In Type Ia instability all four modes participate in the resonance: $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$. This is the instability of a bichromatic wave train to bichromatic disturbances. The corresponding conservation laws and resonance condition take the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & k_1 + k_2 - k_3 \\ l_1 & l_2 & l_3 & l_1 + l_2 - l_3 \end{bmatrix} \begin{bmatrix} |b_1|^2 \\ |b_2|^2 \\ |b_3|^2 \\ |b_4|^2 \end{bmatrix} = \begin{bmatrix} A \\ M^x \\ M^y \end{bmatrix}.$$

Again, the specification of an initial bichromatic wave train consisting of modes $\mathbf{k}_1$, $\mathbf{k}_2$ fixes a particular solution, and the general solution is equivalent to setting $|b_1|^2 = A\eta$, $|b_2|^2 = A\eta$, $|b_3|^2 =$

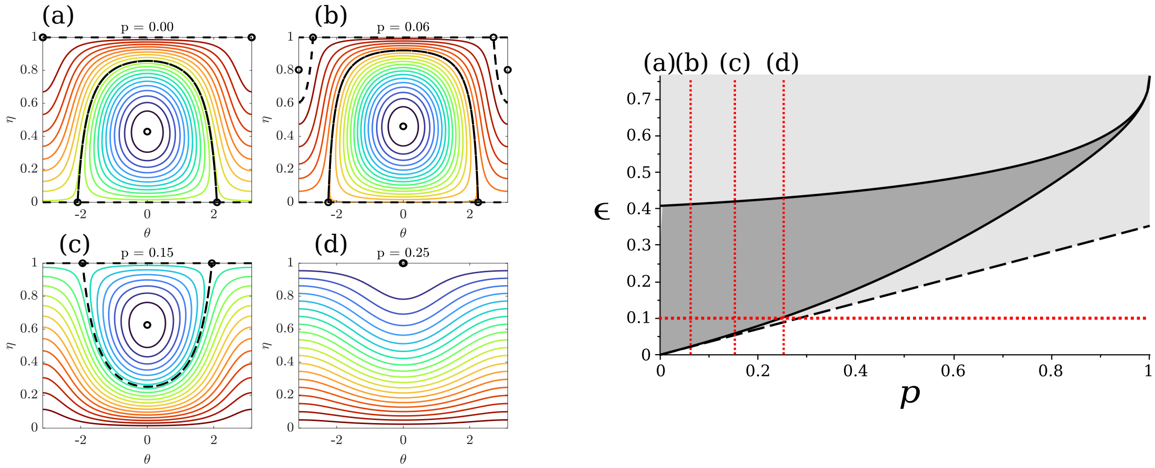

Figure 1: (Left panels) Phase portraits of system (4)–(5) with $D = 1$ with carrier steepness $\epsilon_1 = 0.1$ and varying mode separation p. Dots denote fixed points; thick, dashed lines denote separatrices. (Right panel) Benjamin-Feir instability domains. Dark grey – Zakharov equation. Light grey region – NLS with slope $\epsilon = p/\sqrt{8}$. Dotted red lines show ϵ and p for cases (a)–(d) depicted in left panel.

$A(1/2-\eta)$, $|b_4|^2 = A(1/2-\eta)$, corresponding to equipartition of energy among the modes. Once again, in terms of the single amplitude parameter η and dynamic phase $\theta = \phi_1 + \phi_2 - \phi_3 - \phi_4$ we have the planar system

$$\frac{d\eta}{dt} = 2AT_{1234}\eta(1-2\eta)\sin(\theta), \tag{6}$$

$$\frac{d\theta}{dt} = \Delta_{12}^{34} + A\Omega_0 + A\Omega_1\eta + 2AT_{1234}(1-4\eta)\cos(\theta), \tag{7}$$

with $\Gamma_1 + \Gamma_2 - \Gamma_3 - \Gamma_4 = A\Omega_0 + A\Omega_1\eta$ and $\Delta_{12}^{34} = \omega_1 + \omega_2 - \omega_3 - \omega_4$.

3 Benjamin-Feir dynamics in the phase plane

The best known instability in the context of deep-water waves is the Benjamin-Feir or modulational instability, which will be illustrated in some detail – the general Type Ib instability adds a non-resonant satellite, and Type Ia instabilities begin with a bichromatic basic state, but otherwise the analysis is similar. The entire evolution of a degenerate quartet of water waves $\mathbf{k}_1 = \mathbf{k}_3 + \mathbf{k}_4$ can be described by the compact system (4)–(5) with $B = 0$. It is easy to see that fixed points of this system can only occur in four types: at $\eta = 0$ or 1, with corresponding values of θ, and at $\theta = 0$ or $\pm\pi$ with corresponding values of η.

We relate $A = |B_1|^2 = 2ga^2\pi^2\omega_1^{-1} = 2\pi^2 g^{1/2}\epsilon^2 k_1^{-5/2}$ for a single wave, so that the wave action A is related to the carrier steepness ϵ. The phase-plane is the truncated cylinder $\{(\theta,\eta) \mid -\pi \le \theta \le \pi,\ 0 \le \eta \le 1\}$, whose top $\eta = 1$ are the Stokes' waves, and whose bottom $\eta = 0$ are bichromatic wave train solutions of Zakharov's equaion. For unidirectional waves $\mathbf{k}_1 = (1,0)$, $\mathbf{k}_3 = (1+p,0), \mathbf{k}_4 = (1-p,0)$ the mode separation p takes on the role of the sole natural bifurcation parameter.

Some algebra shows that the criterion for existence of fixed points at $\eta = 1$ is equivalent to the linear discriminant criterion for instability [4, Eq. (14.9.16)]. This is shown in figure 1, which also

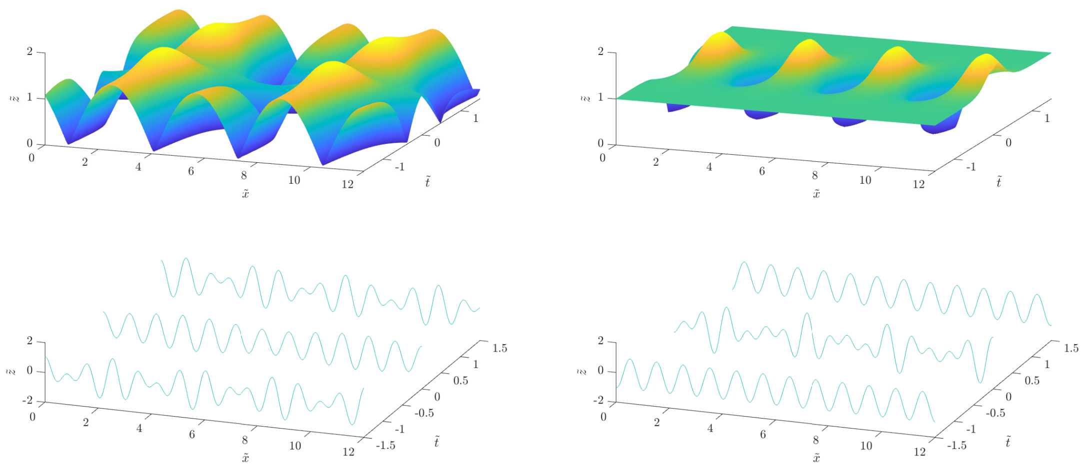

Figure 2: Discrete breather solutions of the BFI system (4)–(5). (Left panels) separatrix connecting fixed points at $\eta = 0$ (multibreather) showing envelope (top) and free surface (bottom). (Right panels) separatrix connecting fixed points at $\eta = 1$ (Akhmediev-type breather with plane wave background) showing envelope (top) and free surface (bottom). The ˜ denotes scaled coordinates.

depicts several characteristic phase portraits for various values of mode separation p. Cases $(a)-(c)$ are unstable (fixed points at $\eta = 0$), while case (d) is stable and exhibits little energy exchange. It is also clear that the largest energy exchange (change in η) coincides with phase coherence (little change in θ) and vice versa, as discussed by [6].

Two key features are also easily visible from the phase portraits: the existence of interior fixed points corresponding to nearly-resonant steady-state degenerate quartets (see recent discussions in [7]), and the separatrices connecting these fixed points. These separatrices are discrete breather solutions which arise from a monochromatic or bichromatic background and exhibit one modulation in time, akin to the Akhmediev breather solution [8] (see figure 2). Such breathers are a generic feature of both Type Ia and Ib instability, and arise naturally from the attendant dynamics.

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