The Cauchy-Poisson problem for a fluid with a shear flow and non-uniform compressed ice cover

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1 INTRODUCTION

The study of the unsteady motion of a heavy fluid arising due to the action at the initial time of impulse pressure or an elevation of the horizontal equilibrium surface of the fluid (the so-called Cauchy - Poisson problem) is one of the classical problems of wave hydrodynamics, which has been studied in some detail for the case of a homogeneous incompressible inviscid fluid, when its upper boundary is free, and the fluid is initially at rest (see, e.g., the famous review of Wehausen and Laitone, 1960 [1]).

In recent years, more complex formulations of this problem have been actively studied. The axisymmetric initial disturbance created on the surface of a fluid under an ice cover without taking into account the compressive forces was carried out by Maiti and Mandal (2005) [2]. In the initial state, the fluid is at rest and its depth is infinite. The initial disturbances were either in the form of an impulsive pressure distributed over a certain region of the ice-cover or an initial displacement of the ice-cover.

The 3-D Cauchy-Poisson problem for a fluid with a shear flow and a surface tension on the free surface was studied by Ellingsen (2014) [3]. In an unperturbed state, the linear shear flow is given for only one component of the horizontal fluid velocity. Li and Ellingsen (2015) [4] considered a similar problem without taking into account surface tension for the case of an initial pressure impulse condition. The results obtained showed that a shear current has a significant impact on the transient wave motions. The shear current introduced asymmetry between upstream and downstream waves, resulting in asymmetric wave patterns. Experimental observations performed by Smeltzer et al. (2019) [5] confirmed the effects predicted by the theory.

This paper presents a solution to a linear 3-D unsteady problem of hydroelasticity on the generation of water waves due to prescribed initial axisymmetric disturbance in the ocean with an ice-cover modelled as a thin elastic plate. For an ice cover, longitudinal, transverse and shear compressive forces are taken into account. Beneath the upper boundary of the fluid, there is a linear shear current with two horizontal velocity components. The integral representation of solution that describes the behavior of the ice cover is constructed. The 2-D case of this problem is considered by Sturova (2022) [6].

2 MATHEMATICAL FORMULATION

We consider an infinitely extended ice cover of constant thickness h and density ρ_1 floating on the surface of an ideal incompressible fluid of depth H. The Cartesian coordinate system x, y, z is introduced so that the x and y axes lie on the unperturbed horizontal upper boundary of the fluid, and the z axis is directed vertically upwards. The velocity vector of the unperturbed flow is denoted by $\mathbf{V} = (U(z), V(z), 0)$, where the velocity components along the x and y axes have a linear dependence

$$U(z) = U_0 + \alpha z, \quad V(z) = V_0 + \beta z.$$

At the initial instant of time $t = 0$, the upper boundary of fluid is deviated from the horizontal position. Let us denote the arising perturbations of the fluid velocity by $\mathbf{v} = (u, v, w)$. The development of the subsequent wave motion in the fluid is described by the linearized Euler equations

$$\Big(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\Big)\mathbf{v} + w\frac{d\mathbf{V}}{dz} + \frac{1}{\rho}\nabla p = 0, \quad \mathrm{div}\ \mathbf{v} = 0, \tag{1}$$

where $p(x, y, z, t)$ is the hydrodynamic pressure, ρ is the fluid density.

The assumption of no cavitation between the ice and fluid at any time gives kinematic and dynamic conditions at $z = 0$:

$$\frac{\partial \eta}{\partial t} + U_0 \frac{\partial \eta}{\partial x} + V_0 \frac{\partial \eta}{\partial y} = w, \tag{2}$$

$$D\Delta_2^2 \eta + Q_1 \frac{\partial^2 \eta}{\partial x^2} + Q_2 \frac{\partial^2 \eta}{\partial y^2} + 2Q_3 \frac{\partial^2 \eta}{\partial x \partial y} + M \frac{\partial^2 \eta}{\partial t^2} + \rho g \eta = p, \quad \Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3}$$

Here, $\eta(x, y, t)$ is the vertical deflection of the ice cover, its cylindrical rigidity is equal to $D = Eh^3/[12(1-\nu^2)]$, $M = \rho_1 h$; E, ν are Young's modulus and Poisson's ratio of ice cover; Q_1, Q_2, Q_3 are longitudinal, transverse and shear forces (compression for positive values and tensile for negative values) along the corresponding directions and g is acceleration due to gravity. It what follows, we will consider the action of only compressive forces, i.e., $Q_j \geq 0$ $(j = 1, 2, 3)$. In the dynamic condition (3), the first term describes the elastic properties of the ice cover, the sum of the next three terms represents the compressive stresses in it, and the fifth term is its inertial properties. In what follows, it is assumed that the inertial term is small in comparison with other terms and can be neglected.

The no-flow condition is satisfied on a flat horizontal bottom

$$w = 0 \quad (z = -H). \tag{4}$$

The initial conditions are as follows:

$$\eta = \eta_0(r), \quad |\mathbf{v}| = 0 \quad (t = 0), \quad r = \sqrt{x^2 + y^2}, \tag{5}$$

where $\eta_0(r)$ is axisymmetric initial elevation of the upper fluid boundary.

To solve the initial-boundary value problem (1)-(5), the double Fourier transform is used

$$\tilde{u}(\lambda, \mu, z, t) = \iint\limits_{-\infty}^{+\infty} u(x, y, z, t) \exp[-i(\lambda x + \mu y)] dx dy. \tag{6}$$

Similar transforms are introduced for other required functions.

Let us perform the Fourier transforms for the system of equations (1) and using the boundary condition (4), we obtain

$$\tilde{w} = A(\lambda, \mu, t) \sinh[k(z + H)], \quad k^2 = \lambda^2 + \mu^2,$$

where $A(\lambda, \mu, t)$ is an undetermined function. Based on the conditions (2) and (3), we have

$$\tilde{p}|_{z=0} = -\frac{\rho}{k} \big\{ \big[\dot{A} + i\zeta_1 A\big] \cosh(kH) - \frac{i\zeta_2}{k} A \sinh(kH) \big\},$$

where a dot denotes a time derivative, $\zeta_1(\lambda, \mu) = \lambda U_0 + \mu V_0$, $\zeta_2(\lambda, \mu) = \lambda\alpha + \mu\beta$. As a result, the equation for determining $\tilde{\eta}(\lambda, \mu, t)$ has the form

$$\ddot{\tilde{\eta}} + i\Big[2\zeta_1 - \frac{\zeta_2}{k} \tanh(kH)\Big] \dot{\tilde{\eta}} + \Big[\Big(\frac{k}{\rho}\zeta_3 + \frac{\zeta_1\zeta_2}{k}\Big) \tanh(kH) - \zeta_1^2\Big] \tilde{\eta} = 0 \tag{7}$$

with initial conditions

$$\tilde{\eta} = \tilde{\eta}_0(k), \quad \dot{\tilde{\eta}} = 0 \quad (t = 0), \tag{8}$$

where $\zeta_3(\lambda, \mu) = Dk^4 - \lambda^2 Q_1 - \mu^2 Q_2 - 2\lambda\mu Q_3 + \rho g$.

Following Ellingsen (2014) [3], we obtain the dispersion relation as the functional relation between the frequency ω and the wave vector $\mathbf{k} = (k\cos\theta, k\sin\theta)$ for flexural gravity waves in the problem under consideration

$$\omega(k, \theta) = \sqrt{\omega_0^2 + \zeta_4^2} - \zeta_4, \tag{9}$$

where

$$\omega_0(k, \theta) = \sqrt{\frac{k}{\rho}\zeta_3 \tanh(kH)}, \quad \zeta_4(k, \theta) = \frac{\zeta_2 \tanh(kH)}{2k}, \quad \theta = \arctan\frac{\mu}{\lambda}.$$

Here the function $\omega_0(k,\theta)$ is the dispersion relation for waves in a fluid without a velocity shear. Taking into account the inertial term in Eq. (3), i.e. for $M \neq 0$, the dispersion relation $\omega_0(k,\theta)$ was given by Bukatov et al. (1991) [7], Bukatov (2017) [8], Sturova (2021) [9]. For the existence of a real value of frequency $\omega(k,\theta)$, it is necessary that the radicand in Eq. (9) be non-negative for all values $0 \le \theta \le 2\pi$. This condition guarantees the stability of the floating plate.

The solution to Eq. (7) with initial conditions (8) has the form

$$\tilde{\eta} = \tilde{\eta}_0 \exp(i\zeta_5 t)\big[\cos(\zeta_6 t) - \frac{i\zeta_5}{\zeta_6}\sin(\zeta_6 t)\big], \quad \zeta_5(k,\theta) = \zeta_4 - \zeta_1, \quad \zeta_6(k,\theta) = \omega + \zeta_4.$$

For initial elevation $\eta_0(r) = a\exp(-br^2)$ after the double Fourier transform (6), we have

$$\tilde{\eta}_0(k) = \frac{\pi a}{b}\exp\Big(-\frac{k^2}{4b}\Big).$$

After performing the inverse Fourier transforms and switching to a moving coordinate system

$$X = x - U_0 t, \ Y = y - V_0 t,$$

we obtain a solution for the vertical deflection of the ice cover

$$\eta(X,Y,t) = \frac{a}{4\pi b}\int_0^{2\pi} d\theta \int_0^{\infty} k\exp\Big(-\frac{k^2}{4b}\Big)\Big[1 + \frac{\gamma(k,\theta)}{\sigma(k,\theta)}\Big]\cos\psi(k,\theta,t)dk, \tag{10}$$

where

$$\gamma(k,\theta) = (\alpha\cos\theta + \beta\sin\theta)\tanh(kH)/2 - k(U_0\cos\theta + V_0\sin\theta),$$

$$\sigma(k,\theta) = \sqrt{\omega_0^2(k,\theta) + [(\alpha\cos\theta + \beta\sin\theta)\tanh(kH)/2]^2},$$

$$\omega_0(k,\theta) = \sqrt{k\tanh(kH)[Dk^4 - Q(\theta)k^2 + g\rho]}, \quad Q(\theta) = Q_1\cos^2\theta + Q_2\sin^2\theta + Q_3\sin(2\theta),$$

$$\psi(k,\theta,t) = (X\cos\theta + Y\sin\theta)k - \omega(k,\theta)t, \quad \omega(k,\theta) = \sigma(k,\theta) - (\alpha\cos\theta + \beta\sin\theta)\tanh(kH)/2.$$

The resulting solution (10) becomes axisymmetric only in the absence of the main flow, i.e. $U(z) = V(z) = 0$, and uniform compression of the ice cover $Q_1 = Q_2, \ Q_3 = 0$:

$$\eta(r,t) = \frac{a}{2b}\int_0^{\infty} k\exp\Big(-\frac{k^2}{4b}\Big)J_0(kr)\cos[\bar{\omega}_0(k)t]dk,$$

where

$$\bar{\omega}_0(k) = \sqrt{k\tanh(kH)(Dk^4 - Q_1k^2 + g\rho)},$$

J_0 is the zeroth order Bessel function of the first kind.

Wave patterns of the ice deflections $\eta(X,Y,t)/a$ are shown in Fig. 1(a-f) at the two time instants: $t\sqrt{g/H} = 5$ (Fig. 1(a,c,e)) and $t\sqrt{g/H} = 10$ (Fig. 1(b,d,f)). The following input data are used:

$$E = 5 \ GPa, \ h = 0.5 \ m, \ \nu = 0.3, \ \rho = 1025 \ kg/m^3, \ H = 20m, \ b = 2/H^2.$$

The values $Q_j = 0$ $(j = 1,2,3)$ are used for Fig. 1(a,b) and the values $(Q_1,Q_2,Q_3) = (1.5,1.2,0.5)\sqrt{g\rho D}$ are used for Fig. 1 (c-f). The values $(\alpha,\beta) = (1,0.6)\sqrt{g/H}, \ (U_0,V_0) = (0.6,0.4)\sqrt{gH}$ are used for Fig. 1 (a,b,e,f) and the values $\alpha = \beta = 0, \ U_0 = V_0 = 0$ are used for Fig. 1 (c,d). For each case, the bold black closed curve shows the shape of the wave front, which is defined as $c(K,\theta)t/H$, where $c(k,\theta) = \omega(k,\theta)/k$ is the phase velocity of flexural gravity wave, the value of the wave number $K = 2\sqrt{b}$, at which the degree of the exponential factor in Eq. (10) is equal to -1, and θ varies from 0 to 2π.

More detailed numerical results will be presented at the Workshop.

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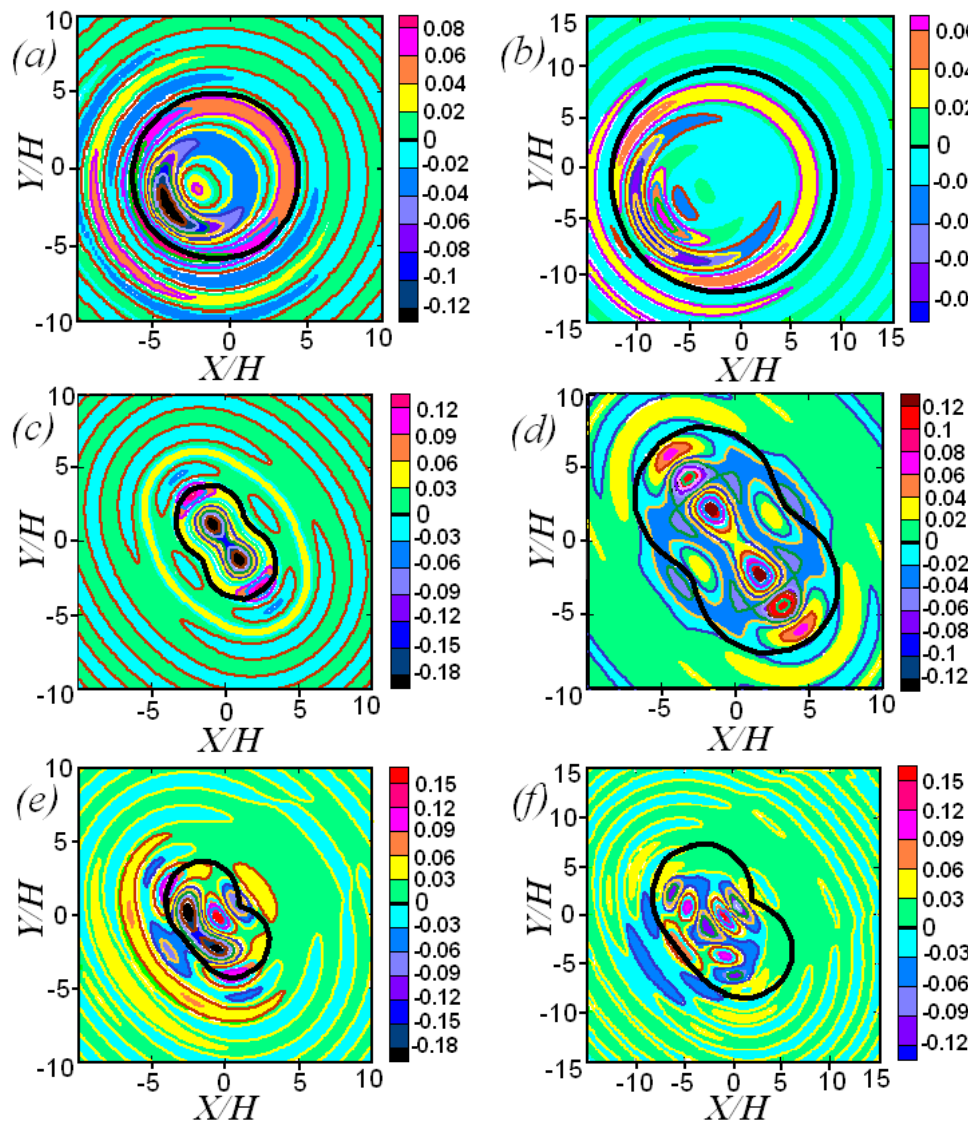

Figure 1.