Steady-state motion of a load on an ice cover with linearly variable thickness in a channel

Konstantin Shishmarev[1,3], Kristina Zavyalova[1] and Tatyana Khabakhpasheva[2]
[1]Department of differential equations, Altai State University, Barnaul, Russia
[2]Lavrentyev Institute of Hydrodynamics SB RAS, Novosibirsk, Russia
[3]E-mail: shishmarev.k@mail.ru

1 INTRODUCTION AND FORMULATION OF THE PROBLEM

The main part of the flexural-gravity wave research was carried out for ice covers of infinite extent. However, majority of the ice tanks, where scientific and technical experiments with ice cover are conducted, have finite dimensions and rectangular cross-sections. For this reason, studying the features of waves in channels and how they differ from waves in unbounded sheets is highly important. Under natural conditions, the ice cover is not homogeneous. Due to different reasons its thickness, density and stiffness are not constant. This paper considers the case of an ice cover whose thickness varies linearly, and ice density and stiffness are assumed to be constant. Similar problem for an ice plate with constant thickness was studied in [1], [2].

The response of a viscoelastic ice cover to a load moving along a frozen channel is considered. The channel is of rectangular section with a finite depth H $(-H < z < 0)$ and a finite width $2b$ $(-b < y < b)$, the channel is of infinite extent in the x direction. Here $Oxyz$ is a Cartesian coordinate system. The problem is studied within the linear theory of hydroelasticity (see, e.g., [3]). Liquid in the channel is inviscid, incompressible and covered with ice. The ice cover is modeled by a thin viscoelastic plate with given constant density ρ_i and rigidity $D(y)$, where $D(y) = Eh_i^3(y)/[12(1-\mu^2)]$, E is the Young's modulus for ice, μ is the Poisson's ratio for ice and $h_i(y)$ is variable thickness of the ice cover. A Kelvin–Voigt model of viscoelastic ice is used in this study. The constitutive equation of this model is $\sigma = E(\epsilon + \tau\partial\epsilon/\partial t)$, where σ is the stress, ϵ is the strain, τ is the so-called retardation time and t is the time. Ice oscillations caused by a load moving along a center line of the channel with constant speed U. The load is modeled by a localized smooth pressure distribution. Flow beneath the plate caused by ice deflections is potential.

The ice deflection $w(x,y,t)$ satisfies the equation of a thin viscoelastic plate

$$M(y)\frac{\partial^2 w}{\partial t^2} + \left(1 + \tau\frac{\partial}{\partial t}\right) Lw = p(x,y,0,t) + P(x,y,t) \quad (-\infty < x < \infty, -b < y < b, z = 0), \quad (1)$$

where L is a differential operator,

$$L = D(y)\Delta^2 + 2D_y\left(\frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial x^2 \partial y}\right) + D_{yy}\left(\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2}\right),$$

$\Delta^2 = \partial^4/\partial x^4 + 2\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4$, $M(y) = \rho_i h_i(y)$ is the mass of the ice plate per unit area, $P(x,y,t)$ is the external pressure and $p(x,y,0,t)$ is a liquid pressure at the ice/liquid interface. The external pressure $P(x,y,t)$ moves along the central line of the channel with constant magnitude P_0 and is described by

$$P(x,y,t) = -P_0 P_1\left((x-Ut)/b\right) P_2\left(y/b\right) \quad (-\infty < x < \infty,\ -b < y < b), \quad (2)$$

$$P_1(\widetilde{x}) = \left(\cos(\pi c_1 \widetilde{x}) + 1\right)/2 \ \ (c_1|\widetilde{x}| < 1), \quad P_1(\widetilde{x}) = 0 \ \ (c_1|\widetilde{x}| \geq 1), \quad \widetilde{x} = (x-Ut)/b,$$

$$P_2(\widetilde{y}) = \big(\cos(\pi c_2 \widetilde{y}) + 1\big)/2 \;\; (c_2|\widetilde{y}| < 1), \quad P_2(\widetilde{y}) = 0 \;\; (c_2|\widetilde{y}| \geq 1), \qquad \widetilde{y} = y/b,$$

where c_1 and c_2 are non-dimensional parameters of the external load characterizing the size of the pressure area. The hydrodynamic pressure $p(x, y, 0, t)$ at the ice-liquid interface is given by the linearised Bernoulli equation,

$$p(x, y, 0, t) = -\rho_\ell \varphi_t - \rho_\ell g w \quad (-\infty < x < \infty, \, -b < y < b), \tag{3}$$

where g is the gravitational acceleration, ρ_ℓ is the density of the liquid and $\varphi(x, y, z, t)$ is the velocity potential of the flow beneath the ice cover. The velocity potential $\varphi(x, y, z, t)$ satisfies Laplace's equation in the flow region and the boundary conditions

$$\varphi_y = 0 \;\; (y = \pm b), \quad \varphi_z = 0 \;\; (z = -H), \quad \varphi_z = w_t \;\; (z = 0). \tag{4}$$

The ice cover is frozen to the walls, which is modeled by the clamped conditions

$$w = 0, \quad w_y = 0 \qquad (-\infty < x < \infty, \;\; y = \pm b). \tag{5}$$

The term with $\tau\partial/\partial t$ in the equation of viscoelastic plate (1) describes the damping of ice plate oscillations, so they decay far away from the moving load, where $|(x - Ut)| \to \infty$.

One case of linear change in the ice thickness is considered: the ice thickness varies symmetrically across the channel, being the smallest at the center of the channel and the largest at the channel walls. The main parameters of the thickness are its average h_*, minimum h_0 and maximum h_1 values. Then $h_i(y)$ can be written in the form:

$$h_i(y) = h_0(1 + \alpha_1|y/b|), \quad \alpha_1 = (h_1 - h_0)/h_0, \quad h_i(0) = h_0, \; h_i(\pm b) = h_1. \tag{6}$$

The formulated problem is solved in non-dimensional variables denoted by tilde. The half-width of the channel b is taken as the length scale, the ratio b/U as the time scale, and the pressure magnitude P_0 as the pressure scale. The non-dimensional depth of the channel H/b is denoted by h. The moving coordinate system $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ with the origin at the centre of the load is introduce by

$$\widetilde{y} = y/b, \quad \widetilde{x} = (x - Ut)/b, \quad \widetilde{z} = z/b, \quad \widetilde{t} = Ut/b.$$

We are concerned with a steady-state solution in the moving coordinate system,

$$w(x, y, t) = w(\widetilde{x}L + Ut, L\widetilde{y}, t) = w_{sc}\, \widetilde{w}(\widetilde{x}, \widetilde{y}),$$

$$\varphi(x, y, t) = \varphi(\widetilde{x}L + Ut, L\widetilde{y}, t) = \varphi_{sc}\, \widetilde{\varphi}(\widetilde{x}, \widetilde{y}, \widetilde{z}),$$

where w_{sc} and φ_{sc} are the scales of the ice deflection and the velocity potential correspondingly. The scales are chosen as $w_{sc} = P_0/(\rho_\ell g)$ and $\varphi_{sc} = (UP_0)/(\rho_\ell g)$. The scale h_{sc} of the ice thickness, $\widetilde{h}_i(\widetilde{y}) = h_i(y)/h_{sc}$, is equal to h_0.

In the non-dimensional variables the problem reads (tildes are omitted further)

$$mh\mathrm{Fr}^2 h_i w_{xx} + \beta\Big(1 - \varepsilon\frac{\partial}{\partial x}\Big)\left[h_i^3 \nabla^4 w + 6h_i^2 h_{i,y}(w_{yyy} + w_{xxy}) + 6h_i h_{i,y}^2 (w_{yy} + \mu w_{xx})\right] + w =$$
$$= h\mathrm{Fr}^2 \varphi_x - P_1(x)P_2(y) \quad (-\infty < x < \infty, \quad -1 < y < 1, \quad z = 0), \tag{7}$$

$$\nabla^2 \varphi = 0 \qquad (-\infty < x < \infty, \;\; -1 < y < 1, \;\; -h < z < 0), \tag{8}$$

$$\varphi_z = -w_x \quad (z = 0), \qquad \varphi_y = 0 \quad (y = \pm 1), \qquad \varphi_z = 0 \quad (z = -h), \tag{9}$$

$$w = 0, \quad w_y = 0 \qquad (y = \pm 1), \qquad w, \varphi \to 0 \qquad (|x| \to \infty). \tag{10}$$

Here $\beta = D_*/(\rho_l g b^4)$, $D_* = Eh_{sc}^3/[12(1-\mu^2)]$, $\varepsilon = (\tau U)/b$, $m = (\rho_i h_{sc})/(\rho_\ell b)$ and $\mathrm{Fr} = U/\sqrt{gH}$ is the Froude number.

The solution of the problem (7) – (10) depends on seven non-dimensional parameters h, m, β, ε, Fr, c_1, c_2 and on the non-dimensional ice thickness $h_i(y)$. These parameters describe the aspect ratio of channel, characteristics of ice and of the applied load. We shall determine the deflection w and strain distribution in the ice cover for given values of these parameters.

METHOD OF THE SOLUTION AND DISCUSSION

The coupled problem (7) – (10) is solved with the help of the Fourier transform in the x direction. The plate equation (7) provides

$$\beta(1-i\xi\varepsilon)\left[h_i^3(w_{yyyy}^F - 2\xi^2 w_{yy}^F + \xi^4 w^F) + 6h_i^2 h_{i,y}(w_{yyy}^F - \xi^2 w_y^F) + 6h_i h_{i,y}^2(w_{yy}^F - \mu\xi^2 w^F)\right] + \\ +(1 - mh\mathrm{Fr}^2\xi^2 h_i)w^F = i\xi h\mathrm{Fr}^2\varphi^F - P^F(\xi, y), \quad (11)$$

where

$$w^F(\xi, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x, y)e^{-i\xi x}dx, \qquad w(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w^F(\xi, y)e^{i\xi x}dx. \quad (12)$$

The Fourier transform is also applied to the rest of the equations in (7) – (10). The solution for the Fourier image of the ice deflections $w^F(\xi, y)$ is sought in the form of an infinite series of normal vibration modes of an elastic beam taking into account the variable thickness of ice. In the linear case, these modes are described by Bessel functions (see, i.e., [4]) and are orthogonal with a weight. After determining the modes, the principal coordinates in the series for the Fourier image of ice deflections are determined. These principal coordinates depend on the Fourier transform parameter ξ and will be complex-valued due to the coefficients in the equation (11). The Fourier image of the flow velocity potential, which is appeared in (11), is determined by the method of separation of variables from the corresponding boundary-value problem taking into account the kinematic condition. In the end, the dimensionless ice deflections $w(x, y)$ are defined as the inverse Fourier transform applied to $w^F(\xi, x)$ (second equation in (12)). Number of the mods is reduced to a finite number $Nmod$. The limits of integration in the inverse Fourier transform are limited and the resulting integrals are calculated numerically for each mode. The convergence of the numerical solution is checked by varying the number of modes and the integration parameters. For a plate of constant thickness, the solution method is the same as the method used in [1].

Calculations of the ice response were carried out for parameters of the problem corresponding to the experimental ice tank at the Sholem Aleichem Amur State University in Birobidzhan (see [5]): $H = 1$ m, $2b = 3$ m, ice thickness in the tank is chosen to be equal to 0.0035 m. The parameters of ice and liquid in the calculations were: $\rho_i = 917$ kg/m^3, $\rho_\ell = 1024$ kg/m^3, $\mu = 0.3$, $E = 4.2 \cdot 10^9$ Pa, $\tau = 0.1$ s. Minimum and maximum values of the ice thickness and speed of the load U change in the calculations. The average thickness h_* in all calculations did not change and is equal to 0.0035 m.

The shape of ice deflections significantly depends on the speed of the load. It is known that there is an infinite number of dispersion relations in a channel and, accordingly, an infinite number of hydroelastic waves possibly propagating along a channel. A hydroelastic wave will propagate from the moving load with its speed if a phase speed equal to the load speed is existed. Long waves propagate behind, and short ones in the front of the load. However, a change in the ice thickness leads to a change in the characteristics of hydroelastic waves and, therefore, the combination of waves propagating from the load can be different. Two minimum phase speeds for the considered channel are shown in Figure 1a. The solid lines show the results for a plate with constant thickness $h_i = 0.0035$ m, the dashed lines show the results for the plate with linear thickness for $h_0 = 0.002$ m, and the dotted lines show the results for the plate with linear thickness for $h_0 = 0.001$ m. For example, at a load speed $U = 2$ m/s, the first wave (for the first minimum phase speed) always propagates from the load, but the presence of the second wave depends on the change in the ice thickness. In the presented case, such a wave exists only for $h_0 = 0.001$ m.

Dimensionless ice deflections along the center line of the channel are shown in Figure 1b. The solid thick line shows the ice deflections for a plate with constant thickness $h_* = 0.0035$ m, the thin solid line – for a plate with variable thickness $h_0 = 0.002$ m and the dotted line – for $h_0 = 0.001$ m. Waves in front of the load are not observed due to the strong damping effect. Amplitude of the waves behind the load increases together with an increase in the change in the ice thickness. It is seen that the length of the observed waves propagating behind the load increases and the effect of damping decreases.

The main focus of the study is on the effect of the non-uniform ice thickness on the ice deflections and strain distribution. More detailed numerical results will be presented at the Workshop.

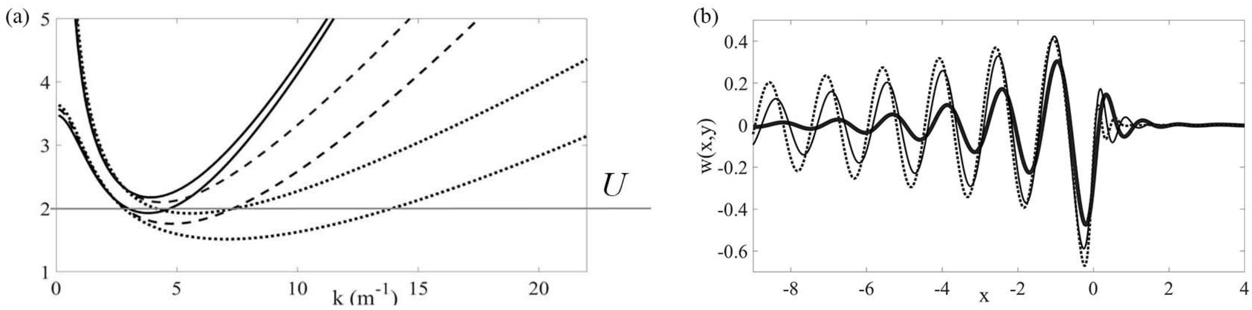

Figure 1: Phase speeds of periodic hydroelastic waves propagating in the channel **(a)**. The ice deflections along the center line of the channel for $U = 2$ m/s **(b)**.

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