Trapped waves within the blocking frequency under compressed sea ice and two-dimensional current

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1. INTRODUCTION

The objective of the current work is to analyze blocking and trapping of flexural-gravity waves that propagate along the coupled water and floating structure surface. As the first step, we need to understand what these waves are and why they are required to be studied. Trapped waves are the waves with finite total energy which exist near the submerged structure and die out as they propagate away from the structure. On the other hand, the wave blocking is a phenomenon in which the rate of wave energy propagation vanishes and is mathematically expressed through linear water theory by the vanishing of the group velocity. The primary cause of such a phenomenon is an opposing current in the ocean which creates a hydrodynamic horizon and blocks the wave. A number of subsequent relevant studies which illustrate the impact of compression and ocean current on wave blocking are available in [1, 2, 3]. In the above-mentioned studies on flexural-gravity wave blocking accounted for the influence of compression but considered only unidirectional ocean currents. The work has been extended to two-dimensional uniform ocean current by [4] where they have also studied the trapped wave for a horizontal submerged circular cylinder. However, in their work, the trapped mode frequency spectrum inside the blocking frequency was not discussed in detail. In the present study , an attempt is made to extend the work of [4] to find the trapped wave mode inside the frequency band of blocking.

2. MATHEMATICAL FORMULATION AND SOLUTION FRAMEWORK

The domain for this physical problem is a homogeneous fluid bounded above by an ice cover and below by a flat and rigid bottom. The Cartesian coordinate framework is chosen so that the z-axis points vertically upward and the xy-plane is horizontal. Hence $z = -h$, $z = f < 0\,(x = 0)$ and $z = 0$, respectively, denote the mean position of the flat bottom, the axis of the cylinder, and the mean ice-covered surface, respectively. The region of the problem under consideration is $-\infty < x, y < \infty$, $-h < z < 0$ with an effect of two-dimensional ocean current acting along the xy-plane. Utilizing the linear water wave theory, the complex velocity potential satisfies the three-dimensional Laplace's equation and takes up the following form in the presence of ocean current

$$\Phi(x, y, z, t) = U_1 x + U_2 y + \phi(x, y, z, t), \tag{1}$$

where U_1 and U_2 are the components of the current along the x- and y-directions, respectively. The upper surface boundary condition is of Neumann type as described in [4]:

$$\left(D\partial_z^4 - Q\partial_z^2 + g + m_0\partial_t^2\right)\frac{\partial \phi}{\partial z} = \left(\partial_t + U_1\partial_x + U_2\partial_y\right)^2\phi \quad \text{on} \quad z = 0, \tag{2}$$

with $D = Ed^3/\{12\rho(1-\nu^2)\}$; $Q = N/\rho$; $m_0 = \rho_p d/\rho$. The ice-cover parameters are E, ν, d, N and ρ_p, which, respectively, denote Young's modulus, Poisson's ratio, thickness, uniform in-plane compressive force and the density. Water density is denoted by ρ and the acceleration due to gravity by g.

The bottom boundary condition due to the flat and rigid bed is

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{on} \quad z = -h. \tag{3}$$

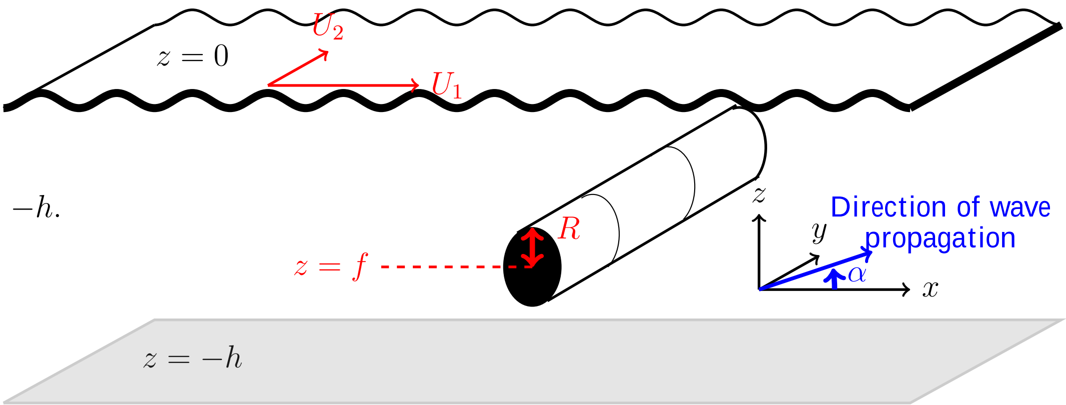

Figure 1: Schematic diagram of the problem (following [4].)

The incident wave angle with the positive x- direction is denoted by α and the angular frequency by ω. The wavenumber of the plane progressive wave that is denoted by k satisfies the following dispersion relation:

$$G(u) = F_+(u)e^{-2uh} - F_-(u) = 0, \tag{4}$$

where

$$F_\pm(u) = u\big(Du^4 - Qu^2 + g\big) \pm \Big[\omega - u(U_1 \cos\alpha + U_2 \sin\alpha)\Big]^2. \tag{5}$$

The multipoles with singularity at the axis of the cylinder satisfying the governing equation (1), and boundary conditions (2) and (3) can be written as

$$\phi_n = \phi_n^a(r,\theta)\cos(by - \omega t) + \phi_n^s(r,\theta)\sin(by - \omega t), \tag{6}$$

with

$$\phi_n^s(r,\theta) = K_n(br)\cos n\theta + \sum_{p=0}^{\infty} d_{pn} I_p(br)\cos p\theta, \tag{7}$$

$$\phi_n^a(r,\theta) = K_n(br)\sin n\theta + \sum_{m=0}^{\infty} c_{mn} I_m(br)\sin m\theta, \tag{8}$$

where $v = b\cosh u$; $I_n(br)$ and $K_n(br)$ are the n-th order modified Bessel functions of the first and second kind, respectively, with argument br; c_{mn} and d_{pn} have the following representations: $c_{mn} = (-1)^m \epsilon_n \int_0^\infty \sinh mu \sinh nu \Big[A(v)e^{vf} + A(v)e^{-v(f+2h)} + e^{-2v(f+h)}\Big]\, du,$
$d_{pn} = (-1)^p \epsilon_n \int_0^\infty \cosh pu \cosh nu \Big[A(v)e^{vf} + A(v)e^{-v(f+2h)} + e^{-2v(f+h)}\Big]\, du,$ with $\epsilon_0 = 1, \epsilon_n = 2, n \geq 1$ and $A(v) = F_+(v)\frac{(-1)^n e^{vf} + e^{-v(f+2h)}}{G(v)}$. The linear combinations of all these symmetric and anti-symmetric multipoles when pass through the body boundary condition on the surface of the cylinder give rise to a system of homogeneous linear equations $A\mathbf{x} = 0$. For numerical computation of trapped modes, we need to truncate the order of the matrix and locate the frequency for which the determinant of the truncated matrix A vanishes.

Before we compute the trapping frequency in the blocking range, it is imperative to look into the integral that arises in c_{mn} and d_{pn}. It has singularity when $G(v) = 0$. Inside the blocking frequency band, except at the terminal frequencies (primary and secondary blocking) and the point of inflexion, there are three distinct positive real roots, i.e., k_1, k_2 and k_3, and each of them induces a singularity for the integral. Consequently, there are three choices of the incident wavenumber, namely k_j ($j = 1, 2, 3$), and we term these waves as the first (for k_1), second (for k_2) and third (for k_3) choice. Therefore, the singular points for the j-th choice are given by

$$L_{ij} = \cosh^{-1}\left[\frac{k_i}{k_j \sin\alpha}\right] \quad \text{for} \quad i = 1, 2, 3,$$

and the corresponding integral can be suitably expressed as the sum of the principal value integrals which can be evaluated by employing the method used in [5]. We will restrict ourselves to first and third choice for evaluating the trapped frequencies.

3 NUMERICAL RESULTS

In this section, the results on trapping phenomena will be discussed in the blocking range when both the currents are acting in opposite direction. Other choices of the currents will be discussed during the presentation. The graphs are shown for the first and third choices of the incident wavenumber. The non-dimensionalized positions of the cylinder and the water depth are kept fixed at $f = -1.01$ and $h = 6$. The flexural rigidity of the floating ice and the compressive force acting on it are kept invariant as $D = 0.01$ and $Q = 1.5\sqrt{D}$.

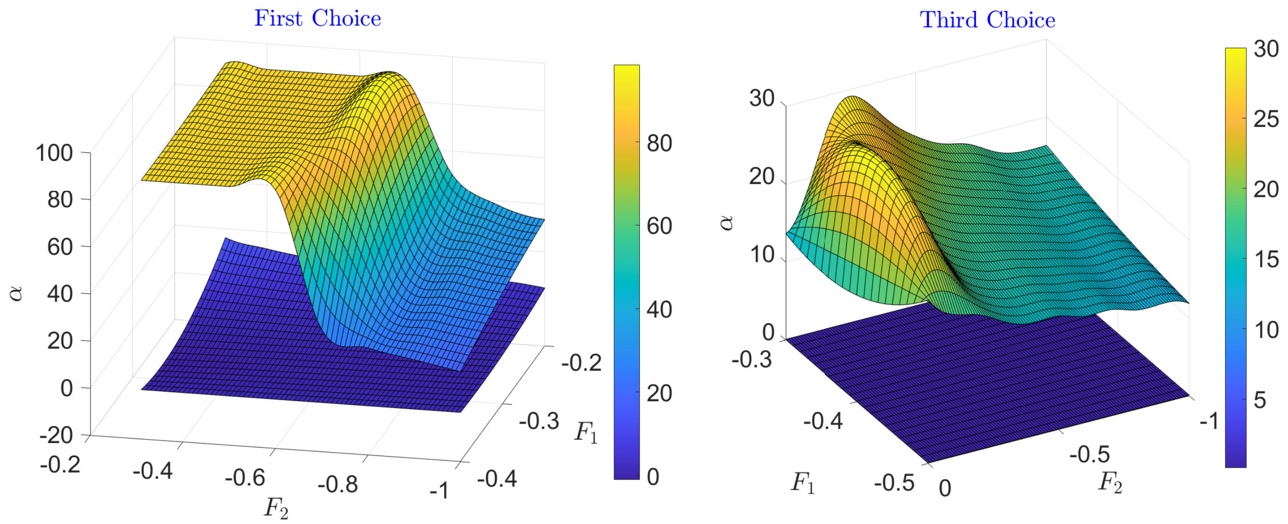

Figure 2: The surface plot of α_{min} and α_{max} for both the choices when the current varies.

The optimum range of α by varying both the opposite currents is shown with the help of surface plots (figure 2). Considering the first choice, the trapped mode exists for $\alpha \in (0, 20^\circ)$ for $F_1 = -0.4$ and $F_2 = -1.0$ (left panel). As we reduce the magnitude of F_2, the range widens and reaches the full range, i.e., $\alpha \in (0, 90^\circ)$. However, with a decrease in the magnitude of F_1, the lower limit of α increases and leads to a narrower α band. Hence, a lower magnitude of F_2 along with a higher magnitude of F_1 is the favourable choice for getting the maximum range of existence of the trapped modes at any value of α. While considering the third choice (right panel), the lower bound of α remains constant at 0° and the upper bound monotonically increases with a decrease in the magnitude of F_1. However, when the magnitude of F_1 decreases, then α_u starts bending and hence obtain maximal value for certain F_2.

Based on the discussion of the range of α in figure 2, the frequency pattern for trapped modes can be understood from figure 3. We fix the value of F_1 at -0.2 and vary F_2 from