Theoretical foundation of a new type of panel method

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Highlights

This abstract summarizes the main features of a new type of panel method, which is based on the Fourier-Kochin method and a weakly-singular boundary-integral flow representation sans waterline integral.

1 Introduction: main features of existing panel methods in marine hydrodynamics

The classical Green function and boundary-integral method implemented in the panel methods used in ship and offshore hydrodynamics are essentially based on applications of Green's fundamental identity

$$\int_{\mathcal{D}} dv\,(\varphi\,\nabla^2_{\boldsymbol{\xi}} G - G\,\nabla^2_{\boldsymbol{\xi}}\varphi) = \int_{\Sigma} da\,\mathbf{m}\cdot(\varphi\,\nabla_{\boldsymbol{\xi}} G - G\,\nabla_{\boldsymbol{\xi}}\varphi) \tag{1}$$

to the flow potential $\varphi(\boldsymbol{\xi})$ and the Green function $G(\boldsymbol{\xi},\mathbf{x})$ that satisfies the free-surface boundary condition relevant to the class of problems under consideration. The unit vector $\mathbf{m}$ in (1) is normal to the closed boundary surface Σ that encloses the flow region $\mathcal{D}$. Although application of Green's basic relation (1) is straightforward in principle, the formulation of integral equations that are well suited for practical applications in ship and offshore hydrodynamics involves nontrivial issues.

In particular, the region where the Green function G in (1) is defined is larger than the flow region $\mathcal{D}$ outside the body surface Σ^H. Specifically, G exists in the lower half space for flows around ships and offshore structures, i.e. in the region outside the hull-surface Σ^H of the body (ship or offshore structure) as well as inside the body. The Green function contains waves that propagate in every direction, including inside the structure, and these waves can result in non-physical waves (eigensolutions) inside the body and spurious solutions for some special wave frequencies, called 'irregular frequencies'. The formulation of a boundary-integral flow representation unaffected by irregular frequencies is nontrivial and indeed has been considered in numerous studies for diffraction-radiation of acoustic or water waves. This issue was recently considered in [1] where a review of the literature related to irregular frequencies may be found and Green's basic identity (1) is applied to a flow-model, called 'rigid-waterplane flow-model', of diffraction-radiation of regular waves by an offshore structure.

Another fundamental difficulty involved in the formulation of a boundary-integral flow representation based on Green's identity (1) is that application of this identity to flows around ships advancing at a constant speed V in calm water or in regular waves of (encounter) frequency ω result in an integral around the mean wetted waterline of the ship. This line integral, called waterline integral hereafter, is a source of notorious difficulties that are well documented in the literature.

Moreover, the Green functions G associated with the free-surface boundary conditions in marine hydrodynamics are defined by singular Fourier integrals and are quite complicated. Indeed, the evaluation of G and $\nabla_{\boldsymbol{\xi}} G$ and the subsequent integration of these functions, which involve complicated singularities, over the hull surface of a ship or offshore structure is a major basic difficulty of panel methods in ship and offshore hydrodynamics.

Lastly, Green's identity (1) involves surface distributions of sources and dipoles associated with the Green function G and its gradient $\nabla_{\boldsymbol{\xi}} G$, which is more singular than G and causes a discontinuity across the dipole distribution.

Thus, there are plenty of good reasons for seeking alternative ways of evaluating potential flows around ships and offshore structures.

2 A weakly-singular boundary-integral flow representation sans waterline integral

An alternative to the classical boundary-integral representation of potential flow around a ship steadily advancing in calm water or in regular waves, which is obtained from Green's identity (1) and involves a waterline integral as was already noted and is well known, is given in [2, 3]. This flow representation is

$$\phi = \int_{\Sigma^H} da\,[G\,q^H + \mathbf{G}\cdot(\mathbf{n}\times\nabla_{\boldsymbol{\xi}}\varphi)] + \int_{\Sigma^F} d\xi\,d\eta\,G\,[q^F - (\mathrm{i}f + F\partial_\xi)\,p^F] \;\text{ where }\; \mathbf{G} \equiv (0, \partial^\xi_\zeta G, -\partial^\xi_\eta G) \tag{2}$$

is a vector Green function that is related to, and easily determined from, the scalar Green function G. The notation ∂^ξ in (2) means integration with respect to ξ. The boundary-integral flow representation (2) provides an integral equation to determine the flow that corresponds to a prescribed flux $q^H \equiv \mathbf{n}\cdot\nabla_{\boldsymbol{\xi}}\varphi$ at the body surface Σ^H and prescribed distributions of pressure p^F and flux q^F at the free surface Σ^F, in accordance with the free-surface boundary condition $\partial_\zeta\varphi+(\mathrm{i}f+F\partial_\xi)^2\varphi = (\mathrm{i}f+F\partial_\xi)\,p^F-q^F$ where $F\equiv V/\sqrt{gL}$ and $f\equiv\omega\sqrt{L/g}$.

The boundary-integral flow representation (2) does not involve a waterline integral. Another notable feature of the flow representation (2) is that it is weakly singular. Indeed, the vector Green function $\mathbf{G}$ defined in (2) is no more singular than the scalar Green function G and the integral over the body surface Σ^H in (2) accordingly is continuous across Σ^H. Lastly, the flow representation (2) is shown in [1, 3] to be free from irregular frequencies.

The boundary-integral flow representation (2) is obtained in [3] via an application of Green's basic identity (1) to the flow-model, called 'rigid-waterplane flow-model', previously used in [1] for diffraction-radiation of regular waves by offshore structures. It is notable and interesting that the boundary-integral flow representation (2), obtained in [3] via Green's classical identity (1) applied to the 'rigid-waterplane flow-model' as was just noted, is identical to the representation of potential flow around a ship steadily advancing in *calm* water obtained in [2] via an analysis based on a *different* flow-model. This flow-model, associated with the theory called Neumann-Michell theory [4, 5], strictly considers the flow outside the ship hull-surface Σ^H but accounts for the (linear) contribution to the integral over Σ^H that stems from the narrow band of water bounded by the undisturbed free surface and its linear approximation.

Moreover, the 'rigid-waterplane flow-model' and the usual 'free-waterplane flow-model' considered in [1] for diffraction-radiation of regular waves by offshore structures are shown in [3] to yield identical boundary-integral flow representations.

That different linear flow-models yield identical flow representations lends credence to the analysis based on the rigid-waterplane flow-model used in [3] for ship motions in regular waves.

Thus, the flow representation (2) holds for the general case $V\omega\neq 0$ as well as for the special cases $\omega=0$ or $V=0$. The flow representation (2) and the consistent expressions for the Green function (associated with optimal Rankine-Fourier decompositions) given in [10] therefore provide a common mathematical basis for the analysis of wave diffraction-radiation by ships and offshore structures.

A main element of the derivation of the flow representation (2) from Green's identity (1) is the identity

$$\int_{\Sigma^H} da\,[(\varphi-\phi)\,\mathbf{n}\cdot\nabla_{\boldsymbol{\xi}}G+(\mathbf{n}\times\nabla_{\boldsymbol{\xi}}\varphi)\cdot\mathbf{G}] = \int_\Gamma d\eta\,(\varphi-\phi)\,\partial_\zeta^\xi G\;, \tag{3}$$

where $\mathbf{G}$ is the vector Green function defined in (2). The identity (3), given in [2, 3], holds for a *general* Green function G, which may be associated with various free-surface boundary conditions, or even boundary conditions associated with other boundary-value problems such as those related to very large floating structures or ice-engineering. Indeed, the identity (3) is general, and can be further generalized. This identity explains how the waterline integral that is obtained if $V\neq 0$ via applications of Green's basic identity (1) can be eliminated.

3 Main features of the Fourier-Kochin method

A core issue in the computation of potential flow around a ship or an offshore structure (body) consists in evaluating flows created by distributions of singularities over the panels (typically flat/curved triangles/quadrilaterals) commonly used to approximate the body surface. This basic issue is considered within the framework of the Fourier-Kochin (FK) method in [6–10]. The FK approach entails two tasks.

The first task consists in evaluating the Kochin (or amplitude) functions, denoted as $\mathcal{A}$ hereafter, associated with a given distribution of singularities. This task, which essentially amounts to integrating the elementary wave function $e^{k\,z+\mathrm{i}\,k\,(x\cos\gamma+y\sin\gamma)}$ over a panel, can be performed easily and efficiently, and indeed is a trivial task that evidently is incomparably simpler than the panel-integration of G and $\nabla_{\boldsymbol{\xi}}G$ that is required in the 'direct Green function method' used in current panel methods. The simplicity of the panel-integration in the FK method in fact is a compelling advantage of that method over the direct Green function method.

However, the second task in the FK method is anything but trivial. This task consists in evaluating singular double Fourier integrals of the general form

$$\phi^F(\mathbf{x}) = \frac{1}{\pi}\int_{-\pi}^{\pi}d\gamma\int_0^\infty dk\,\frac{\mathcal{A}(k,\gamma)\,e^{\mathrm{i}\,k\,(x\cos\gamma+y\sin\gamma)}}{\Delta(k,\gamma)+\mathrm{i}\,\epsilon\,\Delta_1(k,\gamma)}\;\text{ where }\;\epsilon=+0 \tag{4a}$$

and (k,γ) are related to the Cartesian Fourier variables $(\alpha,\beta)=k\,(\cos\gamma,\sin\gamma)$. Moreover, $\Delta(k,\gamma)$ and $\Delta_1(k,\gamma)$ denote dispersion functions associated with the class of flows under consideration. In particular, one has $\Delta\equiv(f+F\alpha)^2/k-1$ for flows around ships steadily advancing in regular waves. The singular double Fourier integral

(4a) is considered in [6–10] for *general* dispersion functions Δ and Δ_1, i.e. for a broad class of dispersive media, and a *general* amplitude (Kochin) function $\mathcal{A}$ associated with arbitrary compact distributions of singularities.

The flow potential ϕ^F defined by the singular double Fourier integral (4a) is expressed in [6–10] as

$$\phi^F(\mathbf{x}) = \phi^W(\mathbf{x};C) + \phi^L(\mathbf{x};C) \tag{4b}$$

where ϕ^W is given by a single (one-fold) Fourier integral along the dispersion curve(s) defined by the dispersion relation $\Delta = 0$, and ϕ^L is given by a regular (non-singular) double Fourier integral. Specifically, for a single closed dispersion curve that surrounds the origin $k = 0$ of the Fourier plane (α,β), as for the dispersion curve related to the ring waves created by a ship advancing in regular waves at $V\omega/g < 1/4$), the component ϕ^W in the fundamental decomposition (4b) is given by the single Fourier integral

$$\begin{aligned} &\phi^W(\mathbf{x}) = \mathrm{i}\int_{-\pi}^{\pi} d\gamma\, \Theta\, \frac{\mathcal{A}(k_*,\gamma)}{\Delta_k(k_*,\gamma)}\, e^{\mathrm{i}\,k_*\,(x\cos\gamma+y\sin\gamma)} \ \text{ where } \ k_* = k_*(\gamma) \\ &\text{and } \ \Theta \equiv \mathrm{erf}[k_*\,(x\cos\gamma + y\sin\gamma)/C] - \mathrm{sign}[\Delta_1(k_*,\gamma)\,\Delta_k(k_*,\gamma)]\,. \end{aligned} \tag{4c}$$

The component ϕ^L is given by the double Fourier integral

$$\phi^L(\mathbf{x}) = \frac{1}{\pi}\int_{-\pi}^{\pi} d\gamma \int_0^\infty dk\, \mathcal{A}^L(k,\gamma)\, e^{\mathrm{i}\,k\,(x\cos\gamma+y\sin\gamma)} \ \text{ where} \tag{4d}$$

$$\mathcal{A}^L = \frac{\mathcal{A}(k,\gamma)}{\Delta(k,\gamma)} - \frac{\mathcal{A}(k_+^*,\gamma)\,e^{-C^2(k/k_+^*-1)^2/4}}{(k-k_+^*)\,\Delta_k(k_+^*,\gamma)} + \frac{\mathcal{A}(k_-^*,\gamma-\pi)\,e^{-C^2(k/k_-^*+1)^2/4}}{(k+k_-^*)\,\Delta_k(k_-^*,\gamma-\pi)} \ \text{ with } \begin{Bmatrix} k_+^* \equiv k_*(\gamma) \\ k_-^* \equiv k_*(\gamma-\pi) \end{Bmatrix}.$$

In expressions (4c-d), $\Delta_k \equiv \partial_k\Delta(k,\gamma)$ and $k_*(\gamma)$ denotes the root of the dispersion relation $\Delta(k,\gamma) = 0$.

The decomposition (4b), given here for a single closed dispersion curve surrounding the origin $k = 0$ of the Fourier plane, holds for general dispersion functions, which typically define several dispersion curves (as is the case in ship hydrodynamics). The component ϕ^W in (4b) represents waves and is defined by a (one-fold) Fourier integral along every dispersion curve defined by the dispersion relation $\Delta = 0$. The component ϕ^L in (4b) is defined by a double Fourier integral with integrand that is smooth everywhere (notably at the dispersion curve) and is dominant near the origin of the Fourier plane, as is numerically illustrated in [6–10].

The wave component ϕ^W and the component ϕ^L in the fundamental decomposition (4b) both involve a positive real number C, although the sum $\phi^W + \phi^L$ is independent of C in accordance with the fact that the decomposition (4) does not involve approximations, i.e. is *exact*. The decomposition (4b) therefore is not unique, and in fact corresponds to a family of alternative decompositions.

In particular, the classical definition of the *principal value of a singular integral*, as well as the *technique of contour integration* in the complex (Fourier) plane, are shown in [6–10] to correspond to the two special choices $C = \infty$ or $C = 0$ in (4b-d). It is also shown in [6–10] that these two classical choices for C are not optimal. In particular, the component ϕ^L includes waves that cancel out the waves defined by the component ϕ^W in large flow regions if C is chosen too large, and the components ϕ^W and ϕ^L are *not smooth* (although the sum ϕ^F is smooth) if C is chosen as $C = 0$. Indeed, an optimal choice for C in (4b-d) and a corresponding *optimal* waves/local-effect (WL) decomposition in which the wave and local components ϕ^W and ϕ^L are *smooth* and ϕ^L represents a *non-oscillatory local* disturbance is shown in [6–10] to be $C \approx 1/2$. This optimal choice is also justified in [10] via an elementary theoretical consideration.

The WL decomposition (4) and the similar flow representation given in [6–10] for dispersion curves represented in the Cartesian form $\alpha = \alpha^*(\beta)$ hold for *arbitrary dispersion functions* Δ and Δ_1 as was already noted. These general WL decompositions are applied in [6–10] to the specific dispersion functions associated with particular classes of flows in ship and offshore hydrodynamics, specifically diffraction-radiation of regular waves by offshore structures in deep water or in finite water depth, and deep-water flows around ships steadily advancing in calm water or in regular waves. These particular applications yield simple expressions, well suited for numerical evaluation, that can be used for the practical evaluation of flows created by *general* compact distributions of singularities for major classes of flows of interest in marine hydrodynamics.

Another notable feature of the FK method expounfded in [10] is that it is based on an optimal Rankine-Fourier (RF) decomposition $G = G^R + G^F$ of the Green functions G associated with the free-surface boundary conditions relevant to ship and offshore hydrodynamics. Thus, the FK theory expounded in [10] involves two optimal flow decompositions related to the Rankine-Fourier decomposition $G = G^R + G^F$ and the previously-noted choice of the parameter C in (4b-d) in the waves/local-flow (WL) decomposition. Indeed, the RF decomposition and the WL decomposition both are non-unique, and optimal decompositions can then be used.

4 Conclusion: a new type of panel methods

The new boundary-integral flow representation (2), together with the Fourier-Kochin (FK) method for evaluating flows created by general distributions of singularities given in [6–10], provide a mathematical basis for a new type of panel method that offers several compelling advantages over the classical 'direct-Green-function' approach pursued for the past fifty years.

In particular, the boundary-integral flow representation (2) is weakly singular, does not involve a waterline integral and is free from irregular frequencies. Moreover, the FK method circumvents the basic difficulties associated with the numerical evaluation of Green functions G and the subsequent integration of G and ∇G over hull-surface panels. Lastly, the major numerical evaluations involved in the FK method, implemented in the manner expounded in [10], are proportional to the number N of panels, whereas the direct-Green-function method requires evaluation and subsequent panel-integration of N^2 singular functions G and ∇G.

In short, the fundamental results given in [3, 10] and summarized in this abstract lay the foundation of panel methods that are considerably simpler than existing methods (notably because the complexities associated with the evaluation and subsequent panel-integration of Green functions are avoided in the FK method) and more robust (because the notorious difficulties related to the line integral around the waterline of a steadily advancing ship are moot) and moreover are expected to be far more efficient than current panel methods (notably because main computations are proportional to the number of panels).

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