LEVEL-CROSSING CONDITIONING IN SECOND-ORDER WATER WAVES

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1 INTRODUCTION

The present study investigates the effect of second-order nonlinearities on wave statistics conditioned to free-surface upcrossing. An upcrossing event is defined as the free-surface upcrossing a given level (i.e. a given altitude) at a position fixed in the reference frame of the mean flow (i.e. the reference frame where the mean fluid velocity field is zero). Based on a trivariate Edgeworth expansion [1], an analytical approximation is derived for the conditional distribution of a second-order wave variable, given free-surface upcrossing.

This analytical model has been applied to three different wave variables: the vertical component of the fluid velocity, the horizontal component of the fluid velocity (both derived from the second-order velocity potential, considered at the mean water level), and the free-surface slope. For these three wave variables, the analytical approximations of their conditional distributions, given upcrossing, have been compared with numerical results obtained from Monte Carlo simulations of second-order irregular seas. These comparisons are used to assess the applicative scope of the analytical model.

2 EDGEWORTH APPROXIMATION

As a first step, an expression is required for the non-conditioned joint distribution of η, $\dot{\eta}$, and ξ, where η is the sea surface elevation, $\dot{\eta}$ its time derivative, and ξ is a third wave kinematic variable. The free surface elevation η is modeled as a stationary differentiable random process, which implies that its time derivative has a zero mean, $E[\dot{\eta}] = 0$. Without loss of generality, it is also assumed that $E[\eta] = E[\xi] = 0$ (if necessary, the random variables of the considered problem can be redefined such that their means are equal to zero). Besides, in order to make analytical expressions more compact, the triad $(\chi = \eta/\sigma_\eta, \dot{\chi} = \dot{\eta}/\sigma_{\dot{\eta}}, \zeta = \xi/\sigma_\xi)$ is introduced, where σ_η, $\sigma_{\dot{\eta}}$, σ_ξ are the standard deviations of η, $\dot{\eta}$, ξ. An approximation for the non-conditional joint distribution of $(\chi, \dot{\chi}, \zeta)$ can be obtained from the Edgeworth expansion of the related probability density function, truncated to the leading order. This approximation takes the form [2]

$$\begin{aligned}\hat{f}_{\chi,\dot{\chi},\zeta} = & \frac{1}{(2\pi)^{3/2}\sqrt{1-\rho^2-\dot{\rho}^2}} J_3 \\ & \times \Bigg[1 + \frac{1}{6}\left(\lambda_{300}H_{300} + 3\lambda_{201}H_{201} + 3\lambda_{120}H_{120} + 6\lambda_{111}H_{111}\right. \\ & \left. + 3\lambda_{102}H_{102} + \lambda_{030}H_{030} + 3\lambda_{021}H_{021} + 3\lambda_{012}H_{012} + \lambda_{003}H_{003}\right) \Bigg] .\end{aligned} \tag{1}$$

A "hat" symbol has been used in Eq. (1) to differentiate the Edgeworth approximation, $\hat{f}_{\chi,\dot{\chi},\zeta}$, from the exact probability density function, $f_{\chi,\dot{\chi},\zeta}$. The function J_3 is given by:

$$J_3(\chi, \dot{\chi}, \zeta) = \exp\left\{ -\frac{1}{2} \frac{(1-\dot{\rho}^2)\chi^2 + (1-\rho^2)\dot{\chi}^2 + \zeta^2 - 2\rho\chi\zeta - 2\dot{\rho}\dot{\chi}\zeta + 2\rho\dot{\rho}\chi\dot{\chi}}{1-\rho^2-\dot{\rho}^2} \right\} . \tag{2}$$

The coefficients

$$\rho = E[\chi\zeta]\,, \tag{3}$$

$$\dot{\rho} = E[\dot{\chi}\zeta]\,, \tag{4}$$

are the non-conditional correlation coefficients of the pairs (χ,ζ) and $(\dot{\chi},\zeta)$. Note that the variables $(\eta,\dot{\eta})$ and their standardized counterparts $(\chi,\dot{\chi})$, non-conditioned, are uncorrelated. The functions H_{abc} are trivariate Hermite polynomials, defined by:

$$H_{abc}(\chi,\dot{\chi},\zeta) = \frac{(-1)^{a+b+c}}{J_3(\chi,\dot{\chi},\zeta)} \frac{\partial^a}{\partial\chi^a}\frac{\partial^b}{\partial\dot{\chi}^b}\frac{\partial^c}{\partial\zeta^c} J_3(\chi,\dot{\chi},\zeta)\,. \tag{5}$$

As calculations will show, explicit expressions for the third-order (i.e. $a+b+c=3$) Hermite polynomials, appearing in Eq. (1), are not required in the present study. The coefficients λ_{abc} (with $a+b+c=3$) are the third-order cumulants of the approximated probability distribution. They may be expressed as:

$$\lambda_{abc} = E\left[\chi^a\dot{\chi}^b\zeta^c\right]\,,\ \text{for } a+b+c\leq 3\,. \tag{6}$$

In the present context of second-order water waves, the cumulants can be analytically computed from the linear and quadratic transfer functions of the related wave variables (see e.g. Longuet-Higgins 1963 [3]).

3 APPLYING RICE'S FORMULA TO EDGEWORTH'S APPROXIMATION

The conditional distribution of a wave variable ξ, given that the free-surface elevation, η, up-crosses the level ℓ, may be expressed in terms of generalised Rice's formula

$$f_{\xi|\eta\uparrow\ell}(\xi) = \frac{\displaystyle\int_0^{+\infty} \tau' f_{\eta,\dot{\eta},\xi}(\ell,\tau',\xi)\,\mathrm{d}\tau'}{\displaystyle\int_0^{+\infty} \tau' f_{\eta,\dot{\eta}}(\ell,\tau')\,\mathrm{d}\tau'} = \frac{1}{\sigma_\xi}\frac{\displaystyle\int_0^{+\infty} \tau f_{\chi\dot{\chi}\zeta}(\tilde{\ell},\tau,\xi/\sigma_\xi)\,\mathrm{d}\tau}{\displaystyle\int_0^{+\infty} \tau f_{\chi,\dot{\chi}}(\tilde{\ell},\tau)\,\mathrm{d}\tau}\,, \tag{7}$$

where $\tilde{\ell} = \ell/\sigma_\eta$. When the distributions $f_{\chi\dot{\chi}\zeta}$ and $f_{\chi,\dot{\chi}}$ are replaced by their truncated Edgeworth expansions, the following approximation is obtained:

$$\hat{f}_{\xi|\eta\uparrow\ell}(\xi) = \frac{1}{\sigma_\xi}\frac{\displaystyle\int_0^{+\infty} \tau \hat{f}_{\chi\dot{\chi}\zeta}(\tilde{\ell},\tau,\xi/\sigma_\xi)\,\mathrm{d}\tau}{\displaystyle\int_0^{+\infty} \tau \hat{f}_{\chi,\dot{\chi}}(\tilde{\ell},\tau)\,\mathrm{d}\tau}\,, \tag{8}$$

where $\hat{f}_{\chi,\dot{\chi}}$ is the leading-order Edgeworth approximation of the joint probability density function of χ and $\dot{\chi}$. The integral at the denominator of Eq. (8) is a normalisation factor which has been already computed in [4], where Longuet-Higgins investigates the upcrossing frequency of the second-order free surface elevation. It can be expressed in the following form:

$$\int_0^{+\infty} \tau \hat{f}_{\chi,\dot{\chi}}(\tilde{\ell},\tau)\,\mathrm{d}\tau = \frac{1}{2\pi} e^{-\frac{1}{2}\tilde{\ell}^2}\left[1 + \frac{\lambda_{300}}{6}H_3(\tilde{\ell}) + \frac{\lambda_{120}}{2}H_1(\tilde{\ell})\right]\,, \tag{9}$$

where H_3 and H_1 are the third and first univariate probabilist's Hermite polynomials. The computation of the numerator of Eq. (8) is more involved. It may be decomposed as follows:

$$\mathscr{F}(\tilde{\ell},\zeta) = \int_0^{+\infty} \tau \hat{f}_{\chi\dot{\chi}\zeta}(\tilde{\ell},\tau,\zeta)\,\mathrm{d}\tau \tag{10}$$

$$= \alpha_{000}G_1(\tilde{\ell},\zeta) + \sum_{a+b+c=3} \alpha_{abc}\mathscr{I}_{abc}(\tilde{\ell},\zeta) \tag{11}$$

where G_1 is given by

$$G_1(\tilde{\ell},\zeta) = \int_0^{+\infty} \tau J_3(\tilde{\ell},\tau,\zeta)\,\mathrm{d}\tau\,, \tag{12}$$

the functions $\mathscr{I}_{abc}$ (with $a+b+c=3$) are given by

$$\mathscr{I}_{abc}(\tilde{\ell},\zeta) = \int_0^{+\infty} \tau H_{abc}(\tilde{\ell},\tau,\zeta) J_3(\tilde{\ell},\tau,\zeta)\,\mathrm{d}\tau\,. \tag{13}$$

The coefficients α_{000} and α_{abc} ($a+b+c=3$) are numerical factors whose expressions can be identified by substituting Eq. (1) into Eq. (10). Inspired by a method initially proposed in [4],[1] the integrals $\mathscr{I}_{abc}$ may be analytically computed by using a combination of integrations by parts and Hermite polynomial algebra. The detailed steps of the calculation are a bit lengthy and not reproduced here. All calculations done, $\mathscr{F}$ can be expressed in the following form:

$$\begin{aligned}\mathscr{F}(\tilde{\ell},\zeta) =&\Gamma_0(\tilde{\ell},\zeta)G_0(\tilde{\ell},\zeta)+\Gamma_1(\tilde{\ell},\zeta)G_1(\tilde{\ell},\zeta)\\ &+\left[\beta_1 H_{100}(\tilde{\ell},0,\zeta)+\beta_2 H_{010}(\tilde{\ell},0,\zeta)+\beta_3 H_{001}(\tilde{\ell},0,\zeta)\right] J_3(\tilde{\ell},0,\zeta)\,,\end{aligned} \tag{14}$$

where G_0 is defined as

$$G_0(\tilde{\ell},\zeta) = \int_0^{+\infty} J_3(\tilde{\ell},\tau,\zeta)\,\mathrm{d}\tau\,. \tag{15}$$

The first-order trivariate Hermite polynomials, $H_{100}, H_{010}, H_{001}$, can be readily obtained from Eq. (5). The terms β_1, β_2, β_3 are numerical coefficients which can be expressed as a function of the coefficients α_{abc}. Γ_0 and Γ_1 are respectively second-order and third-order bivariate polynomials. The functions G_0 and G_1 can be expressed in closed-form, making use of the Gauss error function. The full explicit expressions of these different terms are a bit lengthy and are not reproduced in the present abstract.

4 ILLUSTRATIVE EXAMPLE

In Section 3, the analytical approximation of the conditional distribution, given free-surface upcrossing, has been derived for any wave variable – noted as ξ above (or ζ in standardized form). In the present section, as an illustrative example, the model is applied to a specific wave variable: the vertical component of the fluid velocity, denoted as w (derived from the velocity potential considered at the mean water level). The assumed sea state is short-crested with a significant wave height $H_s = 4$ m and a peak period $T_p = 10$ s. The water depth is infinite. The conditional distribution of w, given upcrossing of the level ℓ by η, is shown in Figure 1. The results are shown for the linear wave model and the second-order wave model. For the second-order wave model, the Edgeworth-based approximation (calculated by substituting Eqs. 9-14 into Eq. 8) is compared with numerical results obtained from multiple Monte Carlo realizations of the considered sea state. Within the linear wave model (dashed line), w, given upcrossing, follows a Rayleigh distribution which is independent of the actual crossing level, ℓ. When second-order wave nonlinearities are introduced, the conditional distribution of w, given upcrossing, is not of Rayleigh type anymore and it does depend on the altitude of the crossing level. In Figure 1, two different altitudes are considered: $\ell = +H_s/2$ (grey); $-H_s/2$ (black). In the present example, the Edgeworth-based analytical approximation (dotted lines) reproduces quite faithfully the results obtained from the Monte Carlo simulations (solid lines).

In order to better assess the scope of the analytical model, it has been applied to a series of configurations, where the sea state properties and the considered wave variable have been varied. For the sea state, the effects of changes in the wave direction distribution, wave camber and water

[1] In [4], Longuet-Higgins deals with bivariate Hermite polynomials, instead of the trivariate Hermite polynomials appearing in Eq. (13).

depth have been investigated. Regarding the wave variable, in addition to w, the wave slope, and the horizontal component of the fluid velocity (both considered along the average wave direction), have been also considered. In some cases, it has been found that the Edgeworth-based approximation provides results that are less satisfactory than those reported in the example of Fig. 1. The limitations of the Edgeworth-based approximation are partly related to the polynomial nature of the correction (the expression into square brackets in Eq. 1) applied to the Gaussian baseline. It translates to two different kinds of spurious features which may develop in the analytical approximation: (i) spurious oscillatory features; (ii) negative probability densities (which have no mathematical meaning). These limitations will be qualitatively and quantitatively discussed during the workshop.

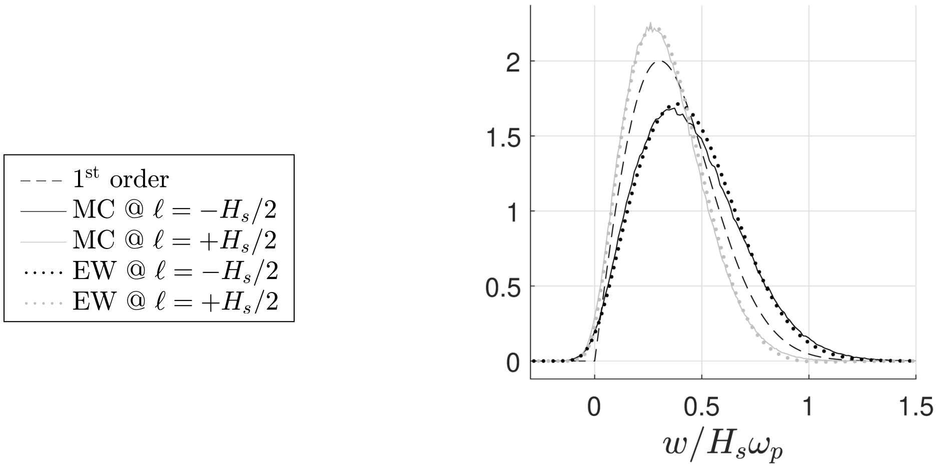

Figure 1: Conditional distribution of w, given free-surface upcrossing. The conditional distribution obtained from the linear wave model (a Rayleigh distribution) is shown as a dashed line (labelled as '1st order' in the legend); it does not depend on the elevation of the crossing level. The conditional distribution obtained from the second-order wave model is shown for two different crossing level elevations: $\ell = -H_s/2$ (black); $+H_s/2$ (grey). Results obtained from the Monte Carlo simulations (labelled as 'MC') are shown as solid lines; the approximations based on Edgeworth expansions (labelled as 'EW') are shown as dotted lines.

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