Derivation of some steady gravity wave equations for uneven bottoms

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1 Introduction

Two-dimensional steady surface waves are fundamental because complex gravity wave fields are often described as interaction between such waves. In constant depth, several efficient numerical algorithms are available and mathematical results (existence, unicity, stability, etc.) are already well-known. The situation is not so advanced for uneven bottoms due to its higher complexity.

A key tool for dealing with water waves is conformal mapping that transforms the (generally unknown) physical domain into a chosen geometry (strip, circle, half-plane, etc.). To the linear approximation, the conformal mapping can be determined independently of the wave field because the upper boundary is flat and the lower boundary is given. This is not the case for the fully nonlinear equations since the shape of the upper boundary is unknown. However, the mapping can be easily obtained as function of the free surface (to be determined) and of the (given) bottom profile. Indeed, for a bottom given by a sufficiently regular function, the latter can be continued to an holomorphic function providing a solution of the conformal mapping, as shown in section 3 below. With this formal solution of the conformal mapping, several equations for the free surface can then be obtained. This is illustrated in section 4 with Byatt-smith [1], Nekrasov [2] and Babenko [3] equations generalised for uneven bottoms of arbitrary (smooth) profile.

2 Physical assumptions, definitions and notations

The fluid is homogeneous, the pressure is zero at the impermeable free surface, while the (uneven) seabed is impermeable. Let be $\{x, y\}$ a Cartesian coordinate system moving with the wave, x being the horizontal coordinate and y the upward vertical one; $y = -d(x)$ and $y = \eta(x)$ denote, respectively, the equations of the bottom and of the free surface. The origin $y = 0$ of the vertical direction is chosen such that η has zero Eulerian average, i.e.,

$$\langle\, \eta \,\rangle \stackrel{\text{def}}{=} \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \eta(x)\, \mathrm{d}x \;=\; 0, \tag{1}$$

hence $y = 0$ is the equation of the mean surface elevation (the still-water level) and $d(x) > 0$. The mean water depth is denoted $\bar{d} \stackrel{\text{def}}{=} \langle d \rangle$ for brevity. The bottom shape $d(x)$ is a given analytic function that can be extended to complex x.

Let be ϕ, ψ, u and v the velocity potential, the stream function, the horizontal and vertical velocities, respectively, such that $u = \partial_x \phi = \partial_y \psi$ and $v = \partial_y \phi = -\partial_x \psi$. The velocity potential satisfies the well-known system of equations

$$\begin{aligned}
\partial_x^2\phi \;+\; \partial_y^2\phi \;&=\; 0 && \text{for} \quad -d(x) \leqslant y \leqslant \eta(x), && (2)\\
\partial_y\phi \;+\; d'\,\partial_x\phi \;&=\; 0 && \text{at} \quad y \;=\; -d(x), && (3)\\
\partial_y\phi \;-\; \eta'\,\partial_x\phi \;&=\; 0 && \text{at} \quad y \;=\; \eta(x), && (4)\\
2\,g\,\eta \;+\; (\partial_x\phi)^2 \;+\; (\partial_y\phi)^2 \;&=\; B && \text{at} \quad y \;=\; \eta(x), && (5)
\end{aligned}$$

where $g > 0$ is the downward constant acceleration of gravity, B is a Bernoulli constant and primes denote the (total) derivative with respect of x. The mean-level condition (1) yields a definition of the Bernoulli constant, i.e., $B = \left\langle u_\mathrm{s}^2 + v_\mathrm{s}^2 \right\rangle$ where, as general notation, subscripts 's' denote the quantities written at the free surface — e.g., $\phi_\mathrm{s}(x) = \phi(x, y = \eta(x))$ — and subscripts 'b' denote the quantities written at the seabed — e.g., $\phi_\mathrm{b}(x) = \phi(x, y = -d(x))$. Note that, for example, $u_\mathrm{s} = (\partial_x \phi)_\mathrm{s} \neq \partial_x(\phi_\mathrm{s}) = u_\mathrm{s} + \eta' v_\mathrm{s}$.

The system (2)–(5) is reduced introducing the complex potential $f \stackrel{\text{def}}{=} \phi + \mathrm{i}\psi$ (with $\mathrm{i}^2 = -1$) and the complex velocity $w \stackrel{\text{def}}{=} u - \mathrm{i}v$ that are holomorphic functions of $z \stackrel{\text{def}}{=} x + \mathrm{i}y$, i.e., $f = f(z)$ and $w = \mathrm{d}f/\mathrm{d}z$. The complex conjugate is denoted with a star (e.g., $z^* = x - \mathrm{i}y$).

The flow being steady, the impermeable free surface and bottom are streamlines, hence ψ_s and ψ_b are constant. A reference velocity is then $c \stackrel{\text{def}}{=} (\psi_\mathrm{b} - \psi_\mathrm{s})/\,đ$. In constant depth, c corresponds to Stokes' second definition of phase velocity, that is the phase velocity observed in the frame of reference without mean flow.

3 Conformal mapping and its formal resolution

The bottom and the free surface being streamlines, it is advantageous to make the change of independent complex variable $z = x + \mathrm{i}y \mapsto \zeta \stackrel{\text{def}}{=} (\mathrm{i}\psi_\mathrm{s} - f)/c$, that conformally maps the fluid domain $x \in \mathbb{R}$ and $y \in [-d(x), \eta(x)]$ into the strip $\alpha \stackrel{\text{def}}{=} \mathrm{Re}(\zeta) \in \mathbb{R}$ and $\beta \stackrel{\text{def}}{=} \mathrm{Im}(\zeta) \in [-đ, 0]$. This conformal mapping yields $c/w = -\mathrm{d}z/\mathrm{d}\zeta$ and the Cauchy–Riemann relations $\partial_\alpha x = \partial_\beta y$ and $\partial_\beta x = -\partial_\alpha y$.

By definition of the conformal mapping, z and ζ are related by the relation [4]

$$z(\zeta) \;=\; x_\mathrm{b}(\zeta + \mathrm{i}đ) \;-\; \mathrm{i}\,d(x_\mathrm{b}(\zeta + \mathrm{i}đ)) \;=\; z_\mathrm{b}(\zeta + \mathrm{i}đ), \tag{6}$$

where $x_\mathrm{b}(\zeta + \mathrm{i}\,đ)$ and $d(x_\mathrm{b}(\zeta + \mathrm{i}đ))$ are analytic continuations of the functions $x_\mathrm{b}(\alpha)$ and $d(x_\mathrm{b}(\alpha))$, respectively. Indeed, at the bottom $\zeta = \alpha - \mathrm{i}đ$, the relation (6) yields

$$z_\mathrm{b}(\alpha) \stackrel{\text{def}}{=} z(\zeta = \alpha - \mathrm{i}đ) \;=\; x_\mathrm{b}(\alpha) \;-\; \mathrm{i}\,d(x_\mathrm{b}(\alpha)) \;=\; z_\mathrm{b}(\alpha), \tag{7}$$

as it should be, while at the free surface $\zeta = \alpha + \mathrm{i}0$, (6) gives

$$z_\mathrm{s}(\alpha) \stackrel{\text{def}}{=} z(\zeta = \alpha) \;=\; x_\mathrm{b}(\alpha + \mathrm{i}đ) \;-\; \mathrm{i}\,d(x_\mathrm{b}(\alpha + \mathrm{i}đ)) \;=\; z_\mathrm{b}(\alpha + \mathrm{i}đ), \tag{8}$$

so once $x_\mathrm{b}(\alpha)$ is known, and $d(x)$ being prescribed, $z(\zeta)$ is known everywhere. Separating real and imaginary parts and introducing Taylor expansions around the bottom, the solution (6) at the free surface yields

$$x_\mathrm{s}(\alpha) \;=\; \cos[đ\,\partial_\alpha]\,x_\mathrm{b}(\alpha) + \sin[đ\,\partial_\alpha]\,d(x_\mathrm{b}(\alpha)), \quad y_\mathrm{s}(\alpha) \;=\; \sin[đ\,\partial_\alpha]\,x_\mathrm{b}(\alpha) - \cos[đ\,\partial_\alpha]\,d(x_\mathrm{b}(\alpha)). \tag{9a,b}$$

Note that, eliminating x_b between equations (9), one gets

$$y_\mathrm{s}(\alpha) \;=\; \tan[đ\,\partial_\alpha]\,x_\mathrm{s}(\alpha) \;-\; \sec[đ\,\partial_\alpha]\,d(\cos[đ\,\partial_\alpha]\,x_\mathrm{s}(\alpha) + \sin[đ\,\partial_\alpha]\,y_\mathrm{s}(\alpha))\,. \tag{10}$$

This expression is explicit for y_s in constant depth but, in general, it is implicit for varying bottoms.

For later convenience, we introduce the self-adjoint positive-definite pseudo-differential operators $\mathscr{C} \stackrel{\text{def}}{=} \partial_\alpha \cot[đ\,\partial_\alpha]$ and $\mathscr{S} \stackrel{\text{def}}{=} \partial_\alpha \csc[đ\,\partial_\alpha]$ that acts on a pure frequency as

$$\mathscr{C}\,\exp(\mathrm{i}k\alpha) \;=\; k\,\coth(k\,đ)\,\exp(\mathrm{i}k\alpha), \qquad \mathscr{S}\,\exp(\mathrm{i}k\alpha) \;=\; k\,\mathrm{csch}(k\,đ)\,\exp(\mathrm{i}k\alpha).$$

Thus, these operators are easily computed and inverted in Fourier space. With these operators, we have in particular

$$\frac{\mathrm{d}\,x_\mathrm{s}}{\mathrm{d}\alpha} \;=\; \mathscr{C}\,y_\mathrm{s} \;+\; \mathscr{S}\,d(x_\mathrm{b})\,, \qquad \frac{\mathrm{d}\,x_\mathrm{b}}{\mathrm{d}\alpha} \;=\; \mathscr{S}\,y_\mathrm{s} \;+\; \mathscr{C}\,d(x_\mathrm{b})\,, \tag{11a,b}$$

relating x to the free surface y_s.

4 Equations for the free surface

Once the conformal mapping has been solved as function of the free surface, an equation for the latter must be derived. Exploiting the holomorphy of different functions, various equations can be obtained. Here, we give generalisations to uneven bottoms of some well-known equations, but many other can be derived. Though not difficult, these derivations are a bit lengthly and thus they are not detailed here. (Similar derivations in constant depth can be found in [5].)

The generalised equations below are written as pseudo-differential equations for simplicity, but their integral formulations are easily obtained expressing the pseudo-differential operators as convolution integrals, e.g., for any function f

$$\mathscr{C}\, f(\alpha) \,=\, \frac{1}{2\,\bar{d}} \fint_{-\infty}^{\infty} \coth\!\left(\frac{\alpha-\gamma}{2\,\bar{d}\,/\,\pi}\right) \frac{\mathrm{d}\,f(\gamma)}{\mathrm{d}\gamma}\, \mathrm{d}\gamma, \qquad \mathscr{C}^{-1}\, f(\alpha) \,=\, -\frac{1}{\pi} \int_{-\infty}^{\infty} \ln\tanh\left|\frac{\alpha-\gamma}{4\,\bar{d}\,/\,\pi}\right| f(\gamma)\, \mathrm{d}\gamma,$$

$$\mathscr{S}\, f(\alpha) \,=\, \frac{\pi}{4\,\bar{d}^2} \int_{-\infty}^{\infty} \operatorname{sech}\!\left(\frac{\alpha-\gamma}{2\,\bar{d}\,/\,\pi}\right)^2 f(\gamma)\, \mathrm{d}\gamma, \qquad \sec[\bar{d}\,\partial_\alpha]\; f(\alpha) \,=\, \frac{1}{2\,\bar{d}} \int_{-\infty}^{\infty} \operatorname{sech}\!\left(\frac{\alpha-\gamma}{2\,\bar{d}\,/\,\pi}\right) f(\gamma)\, \mathrm{d}\gamma.$$

For periodic domains, the corresponding integrals can be easily obtained periodising the kernels.

4.1 Generalised Byatt-Smith equation

The Bernoulli equation at the free surface (5) yields, after conformal mapping and exploiting the relations (10),

$$(\,B \,-\, 2\,g\,y_\mathrm{s}\,) \left\{ [\,\mathscr{C}\, y_\mathrm{s} \,+\, \mathscr{S}\, d(x_\mathrm{b})]^2 \,+\, \left(\frac{\mathrm{d}\,y_\mathrm{s}}{\mathrm{d}\alpha}\right)^2 \right\} \,=\, c^2, \tag{12}$$

together with — from (11b) —

$$x_\mathrm{b} \,=\, \sigma\,\alpha \,+\, \partial_\alpha^{-1}\,[\,\mathscr{S}\, y_\mathrm{s} \,+\, \mathscr{C}\, d(x_\mathrm{b}) \,-\, \sigma\,]\,. \tag{13}$$

with $\sigma \stackrel{\text{def}}{=} \lim_{\Lambda\to\infty} (2\Lambda)^{-1} \int_{-\Lambda}^{\Lambda} (\mathrm{d}x_\mathrm{b}(\alpha)/\mathrm{d}\alpha)\, \mathrm{d}\alpha$ and where ∂_α^{-1} is the integration operator providing an antiderivative with zero average. The equation (12) can be rewritten

$$y_\mathrm{s} \,+\, \sec[\,\bar{d}\,\partial_\alpha]\, d(x_\mathrm{b}) \,=\, \mathscr{C}^{-1} \left[\frac{c^2}{B-2gy_\mathrm{s}} \,-\, \left(\frac{\mathrm{d}\,y_\mathrm{s}}{\mathrm{d}\alpha}\right)^2 \right]^{\frac{1}{2}}, \tag{14}$$

that is the generalisation for uneven bottoms of the equation studied by Byatt-Smith [1] in constant depth.

4.2 Generalised Nekrasov equation

Exploiting the holomorphy of the function $\ell(\zeta) \stackrel{\text{def}}{=} \log(-w/c) = -\log(\mathrm{d}z/\mathrm{d}\zeta)$ and denoting $\theta \stackrel{\text{def}}{=} -\operatorname{Im}\ell$ the inclination angle of a streamline, we have

$$\theta_\mathrm{b}(\alpha) \,=\, -\arctan\!\big(d'(x_\mathrm{b})\big)\,, \qquad \theta_\mathrm{s}(\alpha) \,=\, \arctan\!\left(\frac{\mathrm{d}\,y_\mathrm{s}}{\mathrm{d}\alpha} \bigg/ \frac{\mathrm{d}\,x_\mathrm{s}}{\mathrm{d}\alpha}\right), \tag{15a,b}$$

so θ_b is prescribed by the (given) bottom slope and θ_s has to be determined. After some rather lengthly algebra, one obtains

$$\theta_\mathrm{s} \,=\, \sec[\,\bar{d}\,\partial_\alpha]\,\theta_\mathrm{b} \,+\, \mathscr{C}^{-1} \left\{ \frac{g\,c\,\sin(\theta_\mathrm{s})}{K_{\scriptscriptstyle \mathrm{N}} \,-\, 3\,g\,c\,\partial_\alpha^{-1}\,\sin(\theta_\mathrm{s})} \right\}, \tag{16}$$

where θ_b is given by (15a) and with the integration constant

$$K_{\scriptscriptstyle \mathrm{N}} \,=\, \lim_{\Lambda\to\infty} \frac{1}{2\,\Lambda} \int_{-\Lambda}^{\Lambda} \frac{c^3\,\cos(\theta_\mathrm{s})^3}{(\mathrm{d}x_\mathrm{s}/\mathrm{d}\alpha)^3}\, \mathrm{d}\alpha. \tag{17}$$

An equation for x_{b} as function of θ_{s} can be derived from (13), i.e.,

$$x_{\mathrm{b}} \,=\, \sigma\,\alpha \,+\, \partial_\alpha^{-1}\left[\frac{B}{2\,g\,\bar{d}} \,-\, \sigma \,+\, \mathscr{C}\,d(x_{\mathrm{b}}) \,-\, \frac{\mathscr{S}\left(K_{\scriptscriptstyle\mathrm{N}} \,-\, 3\,g\,c\,\partial_\alpha^{-1}\,\sin(\theta_{\mathrm{s}})\right)^{2/3}}{2\,g}\right]. \tag{18}$$

The equations (16) and (18), together with (15a), form a generalisation of the Nekrasov equation [2] for uneven bottoms.

Note that Krasovskii [6] and others, considered such a generalisation but only for periodic (generally sinusoidal) bottoms and/or mild slopes. Our approach here is also valid for non-periodic bottoms and it is not restricted to small variations of the mean depth. Actually, it can be extended to overturning bottoms.

4.3 Generalised Babenko equation

Exploiting the holomorphy of the complex velocity $w(\zeta)$, after some algebra not detailed here, a generalised Babenko equation [3] for y_{s} is obtained as

$$B\,\mathscr{C}\,y_{\mathrm{s}} \,-\, g\,y_{\mathrm{s}}\,\mathscr{C}\,y_{\mathrm{s}} \,-\, \tfrac{1}{2}\,g\,\mathscr{C}\,y_{\mathrm{s}}^2 \,+\, \left(\tfrac{1}{2}\,B \,-\, g\,y_{\mathrm{s}}\right)\mathscr{S}\,d(x_{\mathrm{b}}) \,=\, K_{\scriptscriptstyle\mathrm{B}} \,-\, \tfrac{1}{2}\,c\,\partial_\alpha^{-1}\,\mathscr{S}\,v_{\mathrm{b}}, \tag{19}$$

together with

$$v_{\mathrm{b}} \,=\, \frac{c\,d'(x_{\mathrm{b}})}{1+d'(x_{\mathrm{b}})^2}\left(\frac{\mathrm{d}\,x_{\mathrm{b}}}{\mathrm{d}\alpha}\right)^{-1} \,=\, \frac{c\,d'(x_{\mathrm{b}})}{1+d'(x_{\mathrm{b}})^2}\,\frac{1}{\mathscr{S}\,y_{\mathrm{s}} \,+\, \mathscr{C}\,d(x_{\mathrm{b}})}, \tag{20}$$

and the integration constant

$$K_{\scriptscriptstyle\mathrm{B}} \,=\, \lim_{\Lambda\to\infty}\,\frac{B}{2\,\Lambda}\int_{-\Lambda}^{\Lambda}\left[\frac{y_{\mathrm{s}}}{\bar{d}} \,-\, \frac{g\,\bar{d}}{2\,B}\,\frac{y_{\mathrm{s}}^2}{\bar{d}^2} \,+\, \frac{d(x_{\mathrm{b}})}{2\,\bar{d}}\right]\mathrm{d}\alpha, \tag{21}$$

x_{b} being given by (13). In constant depth, a Babenko equation [5] is recovered, that is quadratic in nonlinearities. However, its generalisation to uneven bottoms is highly nonlinear, in general.

Babenko-like equations have been of great help for mathematical proofs [7] and numerical algorithms [5], in infinite and constant depth. Their efficiency for uneven bottoms remain to be demonstrated due to their highly nonlinear nature.

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