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Moving load in an ice channel with a crack

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1. Introduction

The problem of ice-water-structure interaction received considerable attentions in recent years. One of a typical case is that of an air cushion vehicle (ACV) moving in ice covered regions. Comprehensive reviews of a load moving on an infinite ice sheet was given by Squire *et al.* (1996). Ni and Zeng (2019) and Sturova and Tkacheva (2019) studied the problem of a load moving in an ice channel, where the ice sheets on both sides of the channel were treated as unbounded. The problem of a load moving along a frozen channel, where the ice sheet is confined by two vertical walls on both sides, also received attention. Shishmarev *et al.* (2016) and Khabakhpasheva *et al.* (2019) studied the problem of a load moving along a frozen channel with clamped edge conditions. The Kelvin-Voigt model of viscoelastic ice was used for a load moving at constant speed, see Shishmarev *et al.* (2016). They estimated the critical speeds in term of the maximum deflection, which were found to be larger than the first critical speed of the elastic ice in Korobkin *et al* (2014). Khabakhpasheva *et al.* (2019) studied the time-dependent problem of a load moving in ice channel without account for damping in the ice model. Recently, Ren *et al.* (2020) studied the problem of waves propagating in an ice channel with a crack. The critical speeds of waves in this channel were obtained. In the present paper, we study a moving load in an ice channel with a crack.

2. Formulation of the problem and solution procedure

The problem of a load moving in an ice channel with a crack in the ice cover at a constant speed U is considered, as sketched in figure 1. A coordinate system that moves together with the external load along the channel is used. The origin O of the system is located at the center of the load. The z-axis points upwards, opposite to the gravitational acceleration $\vec{g}$. The channel is of rectangular cross section with constant depth H and width $2L$. The channel is of infinite extent in the x-direction. The channel is occupied with liquid of density ρ . The liquid is inviscid and incompressible. The ice sheet is modelled by thin viscoelastic Kelvin–Voigt plate. The ice sheet is of constant thickness h and with rigidity $D = Eh^3[12(1-\nu^2)]$, where E is Young's modulus of ice and ν is Poisson's ratio. The ice sheet floating on the water surface is divided into two pieces by a crack at $y = 0$. The external load is modelled by a localised pressure $P(x, y)$ over the upper surface of the ice sheet. We assume that the problem is stationary in the moving coordinate system.

The deflection of the ice sheet, $z = w(x, y)$, is a solution of the viscoelastic plate equation written in the moving coordinates,

$$D(1-\tau U\frac{\partial}{\partial x})\nabla^4 w + \rho_i h U^2 \frac{\partial^2 w}{\partial x^2} = p(x,y,0) - P(x,y) \qquad (-L<y<L, -\infty<x<\infty, z=0)\,, \tag{1}$$

where $\tau = \eta / E$ is the retardation time, η is the viscosity of the ice, ρ_i is the ice density, $p(x, y,0)$ is the hydrodynamic pressure acting on the lower surface of the ice sheet, $P(x, y)$ is the external pressure. The external pressure $P(x, y)$ is assumed constant in the rectangular of length $2a$ and width $2b$ whose center is located at the origin,

$$P(x,y) = P_0 \quad (|x| \le a,\ |y| \le b)\,. \tag{2}$$

In this problem, the deflection of ice across the channel is symmetric, $w(x,-y) = w(x, y)$. Note that the ice sheet is not continuous at $y = 0$, where the ice edges are free of stresses and shear forces,

$$\left(\frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2}\right)w = 0\,, \qquad \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial y^2} + (2-\nu)\frac{\partial^2}{\partial x^2}\right)w = 0\,. \tag{3}$$

The other edges of ice sheet are clamped to the walls of the channel

$$w = 0\,, \qquad \frac{\partial w}{\partial y} = 0 \qquad (y = \pm L)\,. \tag{4}$$

The hydrodynamic pressure $p(x, y,0)$ acting on the ice/water interface is given by the linearized Bernouli equation,

$$p(x,y,0) = \rho U\frac{\partial \varphi}{\partial x} - \rho g w \qquad (-\infty<x<\infty, -L<y<L)\,, \tag{5}$$

where $\varphi(x, y, z)$ is the velocity potential of the flow beneath the ice sheet, which satisfies the Laplace equation in the flow region,

$$\nabla^2\varphi = 0 \qquad (-\infty < x < \infty, -L < y < L, -H < z < 0)\,, \tag{6}$$

and the boundary conditions,

$$\frac{\partial\varphi}{\partial y} = 0 \quad (y = \pm L)\,, \qquad \frac{\partial\varphi}{\partial z} = 0 \quad (z = -H)\,, \qquad \frac{\partial\varphi}{\partial z} = -U\frac{\partial w}{\partial x} \quad (z = 0)\,, \tag{7}$$

where the third equation is the linearized body boundary condition. Due to the damping of the ice sheet, the hydroelastic waves decay far away from the moving load,

$$w, \varphi \to 0 \quad (|x| \to \infty)\,. \tag{8}$$

In the linear theory of hydroelasticity, the strains in the ice sheet vary linearly through the ice thickness being zero at the middle of the plate thickness. At any location, the maximum strain is achieved at the surface of the ice. We are concerned only with positive strains which correspond to elongation of the ice surface and tensile stresses in the ice. The strain tensor is given by

$$\theta(x, y) = -z\begin{pmatrix} w_{xx} & w_{xy} \\ w_{xy} & w_{yy} \end{pmatrix}, \tag{9}$$

where z is the coordinate across the ice thickness, $-h/2 \le z \le h/2$ .The tensor (9) describes the strain field in the ice sheet. In order to find the maximum strain in the ice sheet we need to find the eigenvalues of the strain tensor at each location. The strains are proportional to the maximum pressure P_0 of the external load within the linear theory. The linear theory of hydroelasticity can be used when $w_x^2 + w_y^2$ is small and the strains in the ice sheet are below the yield strain θ_{cr} of the ice. The yield strength of a material is defined as the strain $\theta = \theta_{cr}$ at which a material begins to deform plastically. The strains in the ice sheet should be below the yield strain of ice, to prevent our viscoelastic model from being unrealistic. Strains greater than the yield strain θ_{cr} are assumed to lead to ice fracture.

3. Solution of the problem

The coupled problem (1)-(8) is solved with the help of the Fourier transform in the x direction. The plate equation (1) provides

$$D(1 - i\xi\tau U)\left(\frac{\partial^4 w^F}{\partial y^4} - 2\xi^2\frac{\partial^2 w^F}{\partial y^2} + \xi^4 w^F\right) - \xi^2\rho_i h U^2 w^F = i\xi\rho U\varphi^F - \rho g w^F - P^F(\xi, y)\,, \tag{10}$$

where

$$w^F(\xi, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x, y)e^{-i\xi x}\mathrm{d}x\,, \qquad P^F(\xi, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} P(x, y)e^{-i\xi x}\mathrm{d}x\,.$$

Since $w(x, y)$ is symmetric in the y direction, so is $w^F(\xi, y)$. In this way, $w^F(\xi, y)$ can be sought as the following superposition,

$$w^F(\xi, y) = \sum_{n=1}^{\infty} F_n(\xi)\psi_n(|y|) \quad (0 < |y| < L)\,, \tag{11}$$

with coefficients $F_n(\xi)$ to be determined. The modes $\psi_n(y)$ are non-trivial solutions of the eigen-value problem

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \xi^2\right)^2 \psi_n = \lambda_n^4\psi_n \quad (0 < y < L), \tag{12a}$$

$$\psi_n''(0) = \nu\xi^2\psi_n(0), \quad \psi_n'''(0) = (2-\nu)\xi^2\psi_n'(0), \quad \psi_n(L) = 0, \quad \psi_n'(L) = 0, \tag{12b}$$

with the corresponding eigen values λ_n. The modes $\psi_n(y)$ are orthonormal,

$$\int_0^L \psi_n(y)\psi_m(y)\mathrm{d}y = \delta_{nm}\,, \tag{13}$$

where $\delta_{nn} = 1$ and $\delta_{nm} = 0$ for $n \ne m$. Substituting (11) and (12) into (10), one has

$$\sum_{n=1}^{\infty} F_n(\xi)[D\lambda_n^4(1 - i\xi\tau U) + \rho g - \rho_i h\xi^2 U^2]\psi_n(y) = i\rho\xi U\varphi^F - P^F(\xi, y)\,. \tag{14}$$

We write $\varphi^F(y, z, \xi)$ as

$$\varphi^F(y, z, \xi) = -i\xi U\sum_{n=1}^{\infty} F_n(\xi)\Phi_n(y, z, \xi)\,, \tag{15}$$

where the potentials Φ_n, $n \ge 1$, are the solutions of the following boundary value problem

$$\Phi_{n,yy}+\Phi_{n,zz}=\xi^2\Phi_n \quad (|y|<L,-H<z<0)\,, \tag{16}$$

$$\Phi_{n,y}=0 \quad (y=\pm L)\,, \qquad \Phi_{n,z}=0 \quad (z=-H)\,, \qquad \Phi_{n,z}=\psi_n(|y|) \quad (z=0)\,. \tag{17}$$

Multiplying both sides of (14) by $\psi_k(y)$ and integrating the result in y from 0 to L using the relation (13) and (15), we find

$$F_n(\xi)[D\lambda_n^4(1-i\xi\tau U)+\rho g-\xi^2\rho_i hU^2]\delta_{nk}-\rho\xi^2U^2\sum_{n=1}^{\infty}F_n(\xi)C_{kn}=-P_k\,, \tag{18}$$

where

$$C_{kn}(\xi)=\int_0^L\Phi_n(y,z,\xi)\psi_k(y)\mathrm{d}y\,, \qquad P_k(\xi)=\int_0^L P^F(\xi,y)\psi_k(y)\mathrm{d}y\,.$$

The system (18) can be written in the matrix form

$$\mathbf{A}\vec{F}=\vec{P}\,, \tag{19}$$

$$\mathbf{A}=\rho\xi^2U^2\mathbf{C}-[D\lambda_n^4(1-i\xi\tau U)+\rho g-\rho_i h\xi^2U^2]\mathbf{I}\,,$$

where $\vec{F}=(F_1,F_2,F_3,......)^T$, $\vec{P}=(P_1,P_2,P_3,......)^T$, $\mathbf{I}$ is the unit matrix and $\mathbf{C}=\{C_{kn}\}_{k,n=1}^{\infty}$. The coefficients $C_{kn}(\xi)$ are calculated analytically for each ξ.

To solve equation (19) we distinguish the real and imaginary parts of the vector $\vec{F}$, $\vec{F}=\vec{F}^R+i\vec{F}^I$. Note that all other vectors and elements of the matrices in (19) are real, which provides the systems of nonhomogeneous equations with respect to $\vec{F}^R$ and $\vec{F}^I$ with symmetric matrices. Thus, equation (19) can be written as

$$[\rho\xi^2U^2\mathbf{C}-(D\lambda_n^4+\rho g-\rho_i h\xi^2U^2)\mathbf{I}]\vec{F}^R-D\lambda_n^4\xi\tau U\mathbf{I}\vec{F}^I=\vec{P}\,, \tag{20a}$$

$$D\lambda_n^4\xi\tau U\mathbf{I}\vec{F}^R+[\rho\xi^2U^2\mathbf{C}-(D\lambda_n^4+\rho g-\rho_i h\xi^2U^2)\mathbf{I}]\vec{F}^I=0\,. \tag{20b}$$

In the present problem, $P_k(\xi)$ is an even function of ξ. It can be shown that $F_n^R(\xi)$ are even and $F_n^I(\xi)$ are odd functions of variable ξ. The deflection $w(x,y)$ is obtained by the inverse Fourier transform

$$w(x,y)=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\psi_n(y)\int_0^{\infty}(F_n^R(\xi)\cos(\xi x)-F_n^I(\xi)\sin(\xi x))\mathrm{d}\xi\,. \tag{21}$$

4. Results

The results of the present study are presented in terms of the ice deflections and strains in the moving coordinate system. Calculations are performed for a freshwater ice with density $\rho_i=917\ \mathrm{kg/m^3}$, Young's modulus $E=4.2\times10^9\ \mathrm{N/m^2}$, Poisson's ratio $\nu=0.3$ and the ice thickness $h=10\ \mathrm{cm}$. The retardation time is $\tau=0.1\ \mathrm{s}$. The half-width L of the channel is 10 m and the water depth H is 5 m. Water density is $\rho=1000\ \mathrm{kg/m^3}$ and the gravity acceleration is $g=9.8\ \mathrm{m/s^2}$. The load (2) is applied over a square area, $a=b=1\ \mathrm{m}$, with $P_0=1000\ \mathrm{Pa}$. The load speed U is varied from 3 m/s to 20 m/s.

For waves propagating in such an ice channel with the crack, the first critical speed of the first even hydroelastic wave is 7.09 m/s, and the second critical speed of the first even hydroelastic wave is 8.66 m/s. The nondimensional minimum of the ice sheet deflections, $\min(w(x,y))$, and the nondimensional maximum of the ice sheet deflections, $\max(w(x,y))$, for different load speed are shown in figure 2, where $w_{sc}=P_0/\rho g$. It is seen that the magnitudes of both $\min(w(x,y))$ and $\max(w(x,y))$ peak at $U=8.65\ \mathrm{m/s}$, which is close to but not equal to the second critical speed of the first even hydroelastic wave. There are also peaks at the first critical speed of the same wave, but they are less pronounced, which could be caused by the relatively large value of the retardation time for the speeds close to the first critical speed. The maximum strain in the ice sheet, $\max|\theta(x,y)|$, and maximum strain along the crack, $\max|\theta(x,\pm0)|$, are shown in figure 3, where $\theta_{sc}=P_0h/(2\rho gL^2)$. The wave form for load moving with 8.65 m/s and 8.66 m/s is shown in figure 4. It is seen that the ice deflections differ significantly for these two speeds, even though their values differ only by 0.01 m/s.

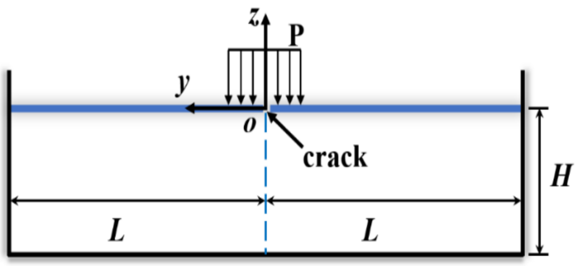

Figure 1. Sketch of a load moving along an ice channel with a crack.

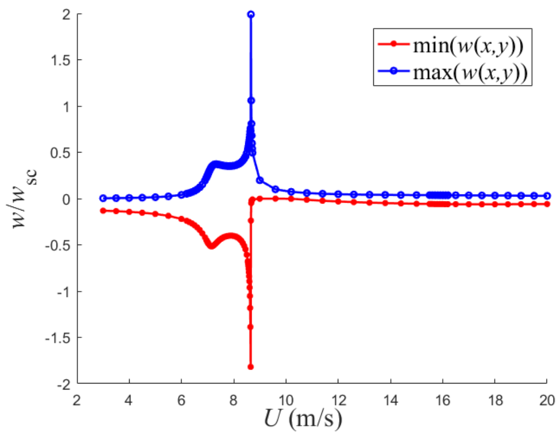

Figure 2 The minimum and maximum ice deflections are shown as functions of the load speed.

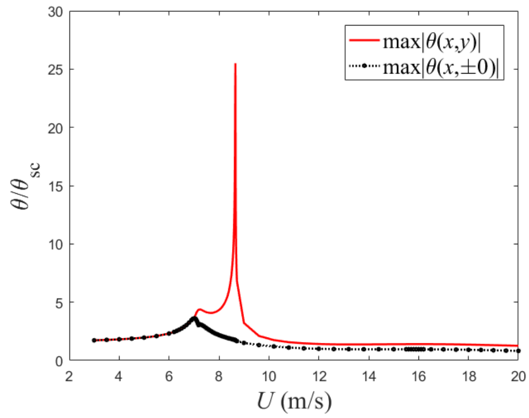

Figure 3 The maximum strains in the whole ice plate and along the crack as functions of the load speed.

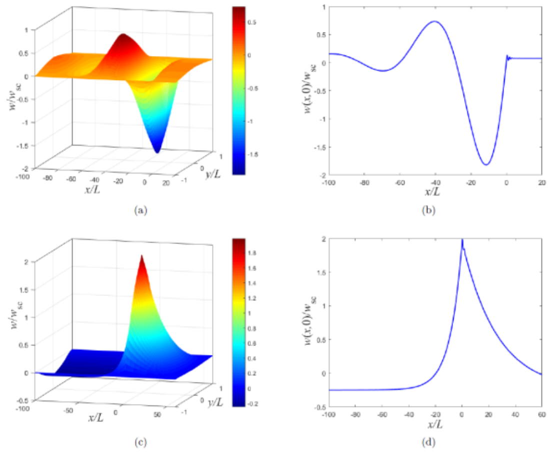

Figure 4 Three-dimensional ice deflections and ice deflection along the crack for load speeds, (a-b) U=8.65 m/s and (c-d) U=8.66 m/s.

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