Higher order phenomena for a two-dimensional breaking wave impact on a vertical wall

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1. Introduction

The scope of this study is to get insight to the unclarified field of the higher order phenomena occurring when a breaking wave hits a vertical wall. Literally, our analysis goes one step further the investigation by Chatjigeorgiou *et al* (2015) and Tsaousis and Chatjigeorgiou (2020). In the first one, the authors propose an innovative approach for the breaking wave impact on a vertical wall, considering two impacted regions and a small air-pocket entrapped. In the latter study, the physical modelling was analogous to the former research, but the expansion of the velocity potential and the free-surface elevation in a perturbation series of t (t is the time) allowed the split of the problem to several orders; the authors solved the first order problem.

Here, a breaking wave idealized in a logical manner, as depicted in Fig. 1, is considered. The wave propagates from right to left with a steady velocity V and hits violently the vertical wall. The water depth is assumed constant and equal to h. During the collision, a small air-pocket, of negligible width ($\delta \to 0$), is entrapped between the water and the wall and extends between $-h \leq z \leq -a$. The hydrodynamic pressure in the air-pocket is assumed zero and no change in its volume is considered. The free-surface elevation at a random time instant, just after impact, is denoted by $H \equiv H(x,t)$.

The novelty of our work is characterized mainly by two particular features: 1) we investigate the higher order phenomena connected to a breaking wave impact on a vertical wall and 2) the higher order boundary value problem reduces to a challenging Sturm-Liouville problem with complicated, infinite series-like boundary conditions, which hold in the two different sections of the wall, i.e., the impacted region and the air-pocket.

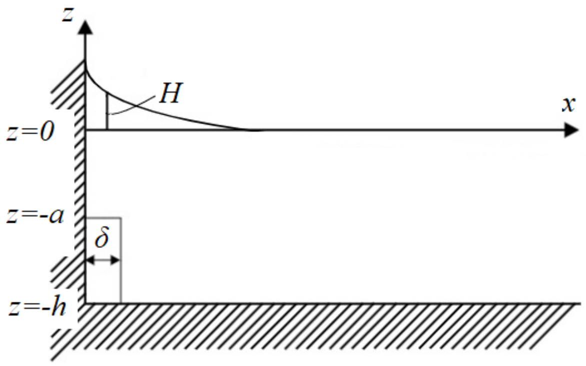

Fig. 1. Schematic representation for the 2D breaking wave impact on a vertical wall

2. The mixed boundary value problem

The problem is considered in its nondimensional form using the following variables

$$\tilde{x} = \frac{x}{h}, \qquad \tilde{z} = \frac{z}{h}, \qquad \tilde{a} = \frac{a}{h}, \qquad \tilde{H} = \frac{H}{h}, \qquad \tilde{\phi} = \frac{\phi}{Vh}, \qquad T = \frac{h}{V}, \qquad \tilde{t} = \frac{t}{T}, \qquad \tilde{p} = \frac{p}{\rho V^2},$$

where a is the upper point of the air pocket, ϕ denotes the velocity potential, t is the time and T is the time scale of the problem, p is the hydrodynamic pressure and ρ is the water density. We investigate the impact in the very early stages where $t \to 0$. We assume inviscid, incompressible fluid and irrotational flow, so that the fluid flow caused by the impact can be described using linear potential theory. The governing set of equations is

$$\nabla^2 \tilde{\phi} = 0, \qquad \tilde{x} \geq 0, \qquad -1 \leq \tilde{z} \leq 0, \tag{1}$$

$$\frac{\partial \tilde{\phi}}{\partial \tilde{x}} = 1, \qquad \tilde{z} \in I_1, \qquad -\tilde{a} < \tilde{z} \leq 0, \qquad \tilde{x} = 0, \tag{2}$$

$$\frac{\partial \tilde{\phi}}{\partial \tilde{t}} = 0, \qquad \tilde{z} \in I_2, \qquad -1 \leq \tilde{z} < -\tilde{a}, \qquad \tilde{x} = 0, \tag{3}$$

$$\frac{\partial \tilde{\phi}}{\partial \tilde{z}} = 0, \qquad \tilde{z} = -1, \qquad \tilde{x} \geq 0, \tag{4}$$

$$\frac{\partial \tilde{H}}{\partial \tilde{t}} = \frac{\partial \tilde{\phi}}{\partial \tilde{z}}, \qquad \tilde{z} = \tilde{H}, \qquad \tilde{x} \geq 0, \tag{5}$$

$$\frac{\partial \tilde{\phi}}{\partial \tilde{t}} + \frac{gh}{V^2}\tilde{H} = 0, \qquad \tilde{z} = \tilde{H}, \qquad \tilde{x} \geq 0, \tag{6}$$

$$\tilde{\phi} \to 0, \qquad \tilde{x} \to \infty, \qquad -1 \leq \tilde{z} \leq 0. \tag{7}$$

In the following we omit the tilde. Conditions (5) and (6) hold on the a priori unknown boundary H. We, therefore, apply a Taylor series expansion in accord with the hydrodynamics of regular surface waves. Hence, Eqs. (5) and (6) become

$$\frac{\partial H}{\partial t} = \frac{\partial \phi}{\partial z} + H\frac{\partial^2 \phi}{\partial z^2} + \cdots, \qquad z = 0, \qquad x \geq 0, \tag{8}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial H}{\partial t}\frac{\partial \phi}{\partial z} + H\frac{\partial^2 \phi}{\partial z \partial t} + \cdots + \frac{gh}{V^2}H = 0, \qquad z = 0, \qquad x \geq 0. \tag{9}$$

Letting t be the small parameter of our perturbation analysis, we write

$$\phi = t\phi_1 + t^3\phi_3 + \cdots, \tag{10}$$

$$H = t^2\eta_2 + t^4\eta_4 + \cdots. \tag{11}$$

Introducing Eqs. (10) and (11) in (8) and (9) and equating terms of like powers of t, we obtain the following higher order mixed boundary value problem (the first order problem is omitted)

$$\nabla^2\phi_3 = 0, \qquad x \geq 0, \qquad -1 \leq z \leq 0, \tag{12}$$

$$\frac{\partial \phi_3}{\partial x} = 0, \qquad z \in I_1, \qquad -a < z \leq 0, \qquad x = 0, \tag{13}$$

$$\phi_3 = 0, \qquad z \in I_2, \qquad -1 \leq z < -a, \qquad x = 0, \tag{14}$$

$$\frac{\partial \phi_3}{\partial z} = 0, \qquad z = -1, \qquad x \geq 0, \tag{15}$$

$$\phi_3 = -\eta_2\frac{\partial \phi_1}{\partial z} - \frac{gh}{3V^2}\eta_2, \qquad z = 0, \qquad x \geq 0, \tag{16}$$

$$\phi_3 \to 0, \qquad x \to \infty, \qquad -1 \leq z \leq 0. \tag{17}$$

The nonlinear wave elevation is obtained from the kinematical condition (8), after equating powers of t^3. This reads

$$\eta_4 = \frac{1}{4}\frac{\partial \phi_3}{\partial z} + \frac{1}{4}\eta_2\frac{\partial^2 \phi_1}{\partial z^2}. \tag{18}$$

The form of the solution that satisfies initially Eqs. (15) and (16), provided that $g'(-1) = 0$ and $g(0) = 1$, is

$$\phi_3 = -f(x)g(z) + \sum_{n=1}^{\infty} F_n(x)\sin(\lambda_n z), \tag{19}$$

where

$$f(x) = \eta_2\frac{\partial \phi_1}{\partial z} + \frac{gh}{3V^2}\eta_2. \tag{20}$$

Further, by exploiting a useful relation from Gradshteyn and Ryzhik (2007) we define

$$g(z) = \frac{4}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\cos[(2i-1)\pi z]}{2i-1}. \tag{21}$$

We introduce Eq. (19) in the Laplace equation and we employ the orthogonality property of $\sin(\lambda_n z)$ in $-1 \le z \le 0$. Moreover, by attempting to satisfy the Neumann and Dirichlet boundary conditions of Eqs. (13) and (14) as well as the far field condition (17), it follows that

$$F_n''(x) - \lambda_n^2 F_n(x) = S_n(x), \qquad 0 \le x < \infty, \tag{22}$$

$$\sum_{n=1}^{\infty} F_n'(0)\sin(\lambda_n z) = f'(0)g(z), \qquad -a < z \le 0, \tag{23}$$

$$\sum_{n=1}^{\infty} F_n(0)\sin(\lambda_n z) = f(0)g(z), \qquad -1 \le z < -a. \tag{24}$$

$$F_n(x) \to 0, \qquad x \to \infty. \tag{25}$$

where

$$S_n(x) = 2\int_{-1}^{0}[f''(x)g(z) + f(x)g''(z)]\sin(\lambda_n z)\,dz. \tag{26}$$

and $\lambda_n = \left(n - \frac{1}{2}\right)\pi$.

Eqs. (22)-(25) determine a novel and challenging one-dimensional, boundary value, Sturm-Liouville problem with mixed conditions. To the authors' best knowledge, relevant problems have not been investigated in the past, at least in the field of analytical hydrodynamics associated with slamming phenomena.

3. The exact solution of the one-dimensional Sturm-Liouville problem

Typically, Sturm-Liouville problems are solved by means of the governing Green's function. The Green's function must satisfy the homogenous form of Eq. (22), must be continuous at $y = x$ and the derivative exhibits a discontinuity (jump singularity) at the same point. For more details, the reader can refer to the book of Chatjigeorgiou (2018) [p. 111]. However, we follow a different approach in this study. A proper solution for the unknown function $F_n(x)$, which encompasses the homogenous and the particular solution, can be written as

$$F_n(x) = A_n e^{-\lambda_n x} - \frac{1}{2\lambda_n}\left\{\int_0^x e^{-\lambda_n(x-\xi)}S_n(\xi)d\xi + \int_x^\infty e^{-\lambda_n(\xi-x)}S_n(\xi)d\xi\right\}, \tag{27}$$

Eq. (27) satisfies the Laplace equation and secures convergence as $x \to \infty$. Substituting Eq. (27) in the boundary conditions (23) and (24) yields the following BVP of mixed type

$$\sum_{n=1}^{\infty}\lambda_n A_n\sin(\lambda_n z) = H_1(z) = -f'(0)g(z) - \frac{1}{2}\sum_{n=1}^{\infty}\int_0^\infty e^{-\lambda_n\xi}S_n(\xi)d\xi\,\sin(\lambda_n z), \qquad -a < z \le 0, \tag{28}$$

$$\sum_{n=1}^{\infty}A_n\sin(\lambda_n z) = H_2(z) = f(0)g(z) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\int_0^\infty e^{-\lambda_n\xi}S_n(\xi)d\xi\,\sin(\lambda_n z), \qquad -1 \le z < -a. \tag{29}$$

The mixed BVP of Eqs. (28) and (29) must be solved in terms of the unknown expansion coefficients A_n. The solution is given explicitly by Sneddon (1966) [p. 158]. However, the authors remarked an inaccuracy in the converge of the infinite series, when following directly the methodology dictated by Sneddon. Nevertheless, an alternative approach for the system of Eqs. (28) and (29) was required.

4. Some numerical results

The calculation of the expansion coefficients A_n for the higher order velocity potential, requires the computation of the first order solution. Nevertheless, a fast convergence of the solution is demonstrated in Fig. 2. An oscillatory behavior around zero is observed for $n \leq 30$, which vanishes thereafter. This, therefore, allows us a proper and careful truncation of the infinite series. In Fig. 3, the higher order velocity potential on the wall for different air-pocket positions is depicted. It is interesting to observe the change in the sign of the potential. This remark manifests that the first order solution overestimates the potential (and consequently the hydrodynamic pressure) in the upper part of the wall, while it underestimates it in a region extending from the half of the height of the wall till the upper point of the air-pocket. The latter is more intense as the air-pocket shrinks. From Fig. 4, the main outcome is that an increase in the impact velocity yields smaller values of the potential. That implies that the first order solution is more accurate for larger impact velocities, yet correction is needed when the impact velocities are rather small. In both cases, one should notice the non-zero value of the potential at the free surface, as prescribed by the condition (16). Finally, it has been observed that the higher order free-surface elevation obtains negative values, which implies that the first order solution overestimates the free-surface elevation.

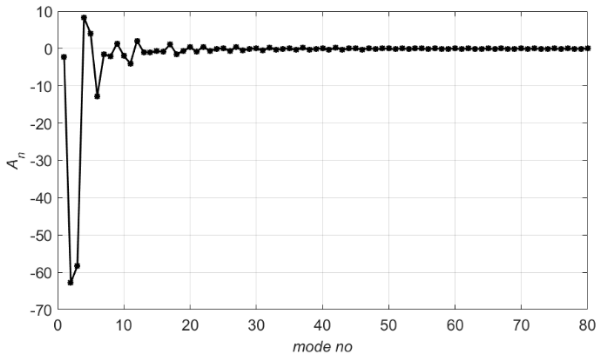

Fig. 2. The expansion coefficients A_n. Here $V = 3$ m/s and the air-pocket extends between $-1 \leq z \leq -0.8$.

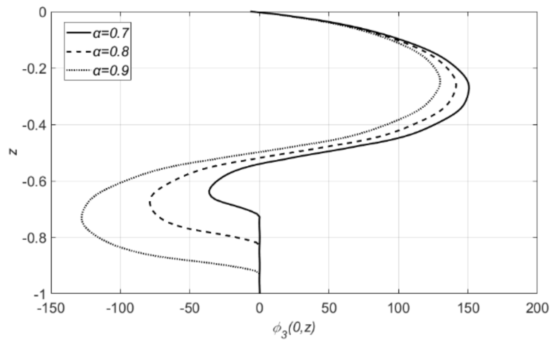

Fig. 3. The higher order velocity potential on the wall, for several air pocket positions. Here, $V = 3$ m/s.

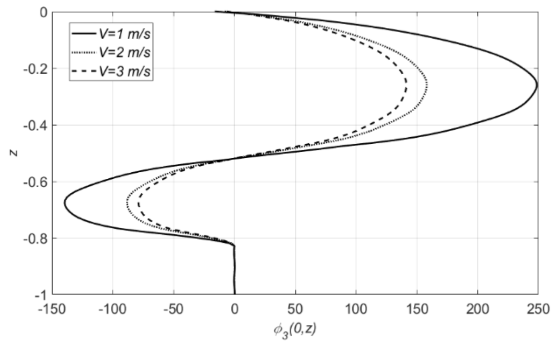

Fig. 4. The higher order velocity potential on the wall, for several impact velocities. Here, the air pocket extends between $-1 \leq z \leq -0.8$.

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References

Chatjigeorgiou IK, Korobkin AA, Cooker MJ, (2015). Two-dimensional breaking wave impact on a vertical wall, In Proceedings of 30th IWWWFB, 12-15 April, Bristol, UK.

Chatjigeorgiou IK, (2018). Analytical Methods in Marine Hydrodynamics, Cambridge University Press: Cambridge, UK.

Gradshteyn IS, Ryzhik IM, (2007). Table of Integrals, Series and Products, 7th edition Elsevier Academic Press, London.

Sneddon, IN, (1966). Mixed Boundary Value Problems in Potential Theory. North-Holland Publishing.

Tsaousis TD, Chatjigeorgiou IK, (2020). An analytical approach for the two-dimensional plunging breaking wave impact on a vertical wall with air entrapment, Fluids, 5, 58.