Moving load on a compressed floating ice cover

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1 INTRODUCTION

At present, the problem of the generation of flexural-gravity waves (FGW), when a localized region of external pressure moves along an ice cover floating on the surface of a fluid, has been sufficiently studied. The moving load might be an ordinary car, a taking off or landing aircraft, or an air-cushion vehicle (ACV). The rectilinear motion of the load was studied in detail in both steady and unsteady cases. A rich bibliography can be found in [1-3]. The ice cover was usually modelled by an initially unstressed, homogeneous isotropic thin elastic plate. However, the ice sheet can experience compression or stretching due to the action of wind, currents or thermal deformations. Three types of compressive forces are possible in 3-D problem: longitudinal, transverse and shear forces [4]. The kinematic properties of FGW arising from periodic and impulse disturbances under conditions of uniform and non-uniform compression forces were studied in detail in [5]. The time-response of a floating flexible plate to the periodic and pulse impact of a distributed load is considered in [6].

This paper presents a solution to a linear 3-D unsteady problem of hydroelasticity on the development of wave motion arising at an instantaneous start and subsequent uniform rectilinear motion of the external load along the ice cover. It is assumed that the load is uniformly distributed in a rectangular area, which simulates the movement of ACV. For an ice cover, longitudinal, transverse and shear compressive forces are taken into account.

2 MATHEMATICAL FORMULATION

We considered an infinitely extended ice cover of constant thickness h and density ρ_1 floating on the surface of an ideal incompressible fluid of depth H. The fluid and the plate is initially unperturbed. Starting from the moment of time $t = 0$, the given localized external pressure acts on the plate, which then moves rectilinearly with constant speed U. The coordinate system x, y, z associated with the moving pressure is introduced, where x and y are horizontal coordinates, and z is a vertical coordinate pointing upwards. The $x-$axis coincides with the direction of load movement. The resulting fluid flow is assumed to be potential, and the velocity of fluid particles and plate deflection are assumed to be small. It is assumed that at all time values the fluid is in contact with the plate.

The velocity potential $\varphi(x, y, z, t)$ satisfies the Laplace equation

$$\Delta\varphi + \partial^2\varphi/\partial z^2 = 0 \quad (|x|, |y| < \infty,\ -H \leq z \leq 0), \quad \Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 \tag{1}$$

with boundary conditions

$$D\Delta^2 w + Q_1\frac{\partial^2 w}{\partial x^2} + Q_2\frac{\partial^2 w}{\partial y^2} + 2Q_3\frac{\partial^2 w}{\partial x \partial y} + M\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 w + \rho g w + \rho\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\varphi = -P(x, y, t) \quad (z = 0), \tag{2}$$

$$\frac{\partial w}{\partial t} - U\frac{\partial w}{\partial x} = \frac{\partial \varphi}{\partial z} \quad (z = 0), \qquad \frac{\partial \varphi}{\partial z} = 0 \quad (z = -H), \tag{3}$$

$$\lim_{r\to\infty} \nabla\varphi = 0 \quad (t \geq 0) \quad (r^2 = x^2 + y^2), \tag{4}$$

and initial conditions

$$\varphi(x, y, z, 0) = w(x, y, 0) = 0. \tag{5}$$

Here, $w(x, y, t)$ is the deflection of the ice cover, $D = Eh_1^3/[12(1-\nu^2)]$, $M = \rho_1 h$; E is Young's modulus, ν is Poisson's ratio of the ice plate; $Q_1,\ Q_2,\ Q_3$ are the longitudinal, transverse and shear compressive forces, respectively; ρ is the fluid density, $P(x, y, t)$ is the external pressure. It is assumed for simplicity that the function $P(x,y,t)$ is nonzero only for $t>0$ in the rectangle of length $2a$ and width $2b$. Inside this rectangle, the pressure is constant

$$P(x,y,t)=P_0 \quad (|x|\le a,\ |y|\le b,\ t>0). \tag{6}$$

Using the Fourier transform, the solution to the problem (1) - (6) has the form

$$w(x,y,t)=-\frac{P_0}{\pi^2}\int_0^\infty B(k)\int_0^{\pi/2} C(k,\theta)\big[G_+(k,\theta,t)+G_-(k,\theta,t)\big]d\theta dk, \tag{7}$$

where

$$G_\pm=\frac{1}{\omega_\pm}\big\{\Lambda_1^\pm\cos[k(x\cos\theta\pm y\sin\theta)]+\Lambda_2^\pm\sin[k(x\cos\theta\pm y\sin\theta)]\big\},$$

$$\Lambda_1^\pm=\frac{1-\cos[(\omega_\pm+kU\cos\theta)t]}{\omega_\pm+kU\cos\theta}+\frac{1-\cos[(\omega_\pm-kU\cos\theta)t]}{\omega_\pm-kU\cos\theta},$$

$$\Lambda_2^\pm=\frac{\sin[(\omega_\pm+kU\cos\theta)t]}{\omega_\pm+kU\cos\theta}-\frac{\sin[(\omega_\pm-kU\cos\theta)t]}{\omega_\pm-kU\cos\theta},\quad \omega_\pm=\omega(k,\pm\theta), \tag{8}$$

$$\omega(k,\theta)=\sqrt{kB(k)[Dk^4-Q(\theta)k^2+\rho g]},\quad B(k)=[Mk+\rho\ \mathrm{cth}(kH)]^{-1}, \tag{9}$$

$$Q(\theta)=Q_1\cos^2\theta+Q_2\sin^2\theta+Q_3\sin(2\theta),\quad C(k,\theta)=\sin(ak\cos\theta)\sin(bk\sin\theta)/(\sin\theta\cos\theta).$$

The function $\omega(k,\theta)$ in Eq. (9) is the dispersion relation for FGW. For the existence of a real frequency value, it is necessary that for all possible values of θ the radicand in Eq. (9) be non-negative. This condition is fulfilled under the condition that $Q(\theta)\le 2\sqrt{g\rho D}$ for $0\le\theta\le\pi$.

The hydrodynamic forces acting on ACV during its movement over the ice cover consist of wave resistance R_x and side force R_y:

$$R_{x,y}(t)=P_0\int_{-a}^{a}\int_{-b}^{b}\Big(\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\Big)dydx. \tag{10}$$

Using the solution (7), we have

$$R_x=\frac{4P_0^2}{\pi^2}\int_0^\infty B(k)\int_0^{\pi/2} C_x(k,\theta)\Big[\frac{\Lambda_2^+}{\omega_+}+\frac{\Lambda_2^-}{\omega_-}\Big]d\theta dk,$$

$$R_y=\frac{4P_0^2}{\pi^2}\int_0^\infty B(k)\int_0^{\pi/2} C_y(k,\theta)\Big[\frac{\Lambda_2^+}{\omega_+}-\frac{\Lambda_2^-}{\omega_-}\Big]d\theta dk,$$

where

$$C_x(k,\theta)=\frac{\sin^2(ak\cos\theta)\sin^2(bk\sin\theta)}{k\sin^2\theta\cos\theta},\quad C_y(k,\theta)=\frac{\sin^2(ak\cos\theta)\sin^2(bk\sin\theta)}{k\sin\theta\cos^2\theta}. \tag{11}$$

According to relation (8), the side force is zero in the absence of shear compressive force in the ice cover since in this case $\omega_+=\omega_-$.

3 STEADY WAVE MOTION

Consider the behavior of wave motion in a steady state, i.e., as $t\to\infty$. In this case, in a moving coordinate system, the problem becomes stationary, with the exception of some values of speed U, called critical [1,2]. For simplicity, we restrict ourselves to considering an infinitely deep fluid ($H\to\infty$). The velocity potential $\varphi(x,y,z)$ satisfies the Laplace equation (1) with the boundary conditions

$$D\Delta^2 w+Q_1\frac{\partial^2 w}{\partial x^2}+Q_2\frac{\partial^2 w}{\partial y^2}+2Q_3\frac{\partial^2 w}{\partial x\partial y}+MU^2\frac{\partial^2 w}{\partial x^2}+\rho g w-\rho U\frac{\partial\varphi}{\partial x}=-P(x,y)\quad (z=0),$$

$$\frac{\partial\varphi}{\partial z}+U\frac{\partial w}{\partial x}=0\quad (z=0),\qquad \frac{\partial\varphi}{\partial z}\to 0\quad (z\to -H).$$

In the far-field zone, the radiation condition is required, which means that waves propagating upstream can be only in the case when their group velocity is greater than the speed of the load, otherwise wave motions exist only downstream.

Using the Fourier transform, we obtain a solution for the deflections of the ice cover

$$w(x,y) = -\frac{2P_0}{\pi^2}\int_0^{\pi/2}\int_0^{\infty}\frac{C(k,\theta)}{k}\Big\{\frac{\cos[k(x\cos\theta+y\sin\theta)]}{Z_+(k,\theta)} + \frac{\cos[k(x\cos\theta-y\sin\theta)]}{Z_-(k,\theta)}\Big\}dkd\theta, \qquad (12)$$

where

$$Z_\pm(k,\theta) = Dk^4 - Q_\pm k^2 + g\rho - kU^2\cos^2\theta(\rho+kM), \quad Q_\pm = Q(\pm\theta).$$

The functions $Z_\pm(k,\theta)$ are the polynomials of the fourth degree in k and for some values of θ they have real positive roots $k_1^\pm(\theta)$ and $k_2^\pm(\theta)$ $(k_1^\pm(\theta) < k_2^\pm(\theta))$. In this case, the inner integral in Eq. (12) is calculated in the sense of the principal value and taking into account the radiation condition in the far field, residues at the singular points should be added. It is well known [1,2] that the group velocity of waves is greater that their phase velocity for shorter flexural waves and less for long gravity-dominated waves. Therefore, for the first term in the inner integral (12), it is necessary to calculate

$$pv\int_0^\infty \frac{C(k,\theta)\cos[k(x\cos\theta+y\sin\theta)]}{kZ_+(k,\theta)}dk$$
$$+\pi\Big\{\frac{C(k_1^+,\theta)\sin[k_1^+(x\cos\theta+y\sin\theta)]}{k_1^+\partial Z_+/\partial k|_{k=k_1^+}} - \frac{C(k_2^+,\theta)\sin[k_2^+(x\cos\theta+y\sin\theta)]}{k_2^+\partial Z_+/\partial k|_{k=k_2^+}}\Big\},$$

where the symbols pv denote the integral in the sense of the principal value. Similar expressions take place for the second term in the inner integral (12) for those values of θ for which the polynomial $Z_-(k,\theta)$ has two positive roots.

It is known [5,7] that there is a critical speed of load motion U_c such that when the load moves with speed $U < U_c$, no wave motions generate in the far-field zone. For infinitely deep fluid, the following relations hold:

$$U_c = \Psi(k_c), \quad d\Psi(k)/dk|_{k=k_c} = 0,$$

where

$$\Psi(k) = \sqrt{\frac{1}{k(\rho+kM)}\Big[Dk^4 - Q_1k^2 + \rho g - \frac{Q_3^2k^4}{Dk^4 - Q_2k^2 + \rho g}\Big]}.$$

The definition of k_c is reduced to calculating the positive root of the 13th degree polynomial.

The wave force acting on the load in the steady problem are calculated by formulas similar to Eq. (10). These forces differ from zero only when the load moves with supercritical speed $U > U_c$ as they are determined by the contributions from residues at singular points of the inner integral in Eq. (12). Let us denote the interval of values of the angular coordinate θ for the singular points of the function $Z_+(k,\theta)$ as $\gamma_1^+ < \theta < \gamma_2^+$, and for the function $Z_-(k,\theta)$ as $\gamma_1^- < \theta < \gamma_2^-$. Then

$$R_x = -\frac{8P_0^2}{\pi}\Big\{\int_{\gamma_1^+}^{\gamma_2^+}\Big[\frac{C_x(k_1^+,\theta)}{k_1^+\partial Z_+/\partial k|_{k=k_1^+}} - \frac{C_x(k_2^+,\theta)}{k_2^+\partial Z_+/\partial k|_{k=k_2^+}}\Big]d\theta$$
$$+\int_{\gamma_1^-}^{\gamma_2^-}\Big[\frac{C_x(k_1^-,\theta)}{k_1^-\partial Z_-/\partial k|_{k=k_1^-}} - \frac{C_x(k_2^-,\theta)}{k_2^-\partial Z_-/\partial k|_{k=k_2^-}}\Big]d\theta\Big\},$$

$$R_y = -\frac{8P_0^2}{\pi}\Big\{\int_{\gamma_1^+}^{\gamma_2^+}\Big[\frac{C_y(k_1^+,\theta)}{k_1^+\partial Z_+/\partial k|_{k=k_1^+}} - \frac{C_y(k_2^+,\theta)}{k_2^+\partial Z_+/\partial k|_{k=k_2^+}}\Big]d\theta$$
$$-\int_{\gamma_1^-}^{\gamma_2^-}\Big[\frac{C_y(k_1^-,\theta)}{k_1^-\partial Z_-/\partial k|_{k=k_1^-}} - \frac{C_y(k_2^-,\theta)}{k_2^-\partial Z_-/\partial k|_{k=k_2^-}}\Big]d\theta\Big\},$$

where the values $C_x(k,\theta),\ C_y(k,\theta)$ in Eq. (11) are used.

When the load moves along the elastic plate at an angle β to the x-axis, the problem should be considered in a new coordinate system $x',\ y',\ z$, where

$$x' = x\cos\beta + y\sin\beta, \quad y' = y\cos\beta - x\sin\beta.$$

In the new coordinate system, the values of the longitudinal, transverse and shear compressive forces are equal to [5,7]

$$Q_1' = Q_1\cos^2\beta + Q_2\sin^2\beta + Q_3\sin(2\beta),$$
$$Q_2' = Q_1\sin^2\beta + Q_2\cos^2\beta - Q_3\sin(2\beta),$$
$$Q_3' = (Q_2 - Q_1)\sin(2\beta)/2 + Q_3\cos(2\beta).$$

The dependence of the critical speed U_c and dimensionless compression parameters $q_j \equiv Q_j/\sqrt{g\rho D}$ $(j = 1,2,3)$ on the angle of rotation β is shown in Fig. 1(a). Curves 1 and 2 correspond to the values U_c at $h = 1m$ and $h = 2m$, respectively. The following input data are used: $E = 5\cdot 10^9$ Pa, $\rho = 1025\text{kg/m}^3$, $\rho_1 = 922.5\text{kg/m}^3$, $\nu = 0.3$, $(q_1,q_2,q_3) = (1.5, 1.2, 0.6)$, $P_0 = 10^3$ Pa, $a = 20$m, $b = 10$m. The dimensionless values of wave resistance $\bar{R}_x$ and side force $\bar{R}_y$ depending on the load speed U for the steady problem is shown at $h = 1m$ in Fig. 1 (b),(c), respectively, where

$$(\bar{R}_x, \bar{R}_y) = \frac{g\rho}{2bP_0^2}(R_x, R_y).$$

Curves 1-4 correspond to the values of the angle $\beta = 0,\ 45,\ 90,\ 135°$. Curve 5 shows the wave resistance at zero values of the compression parameters.

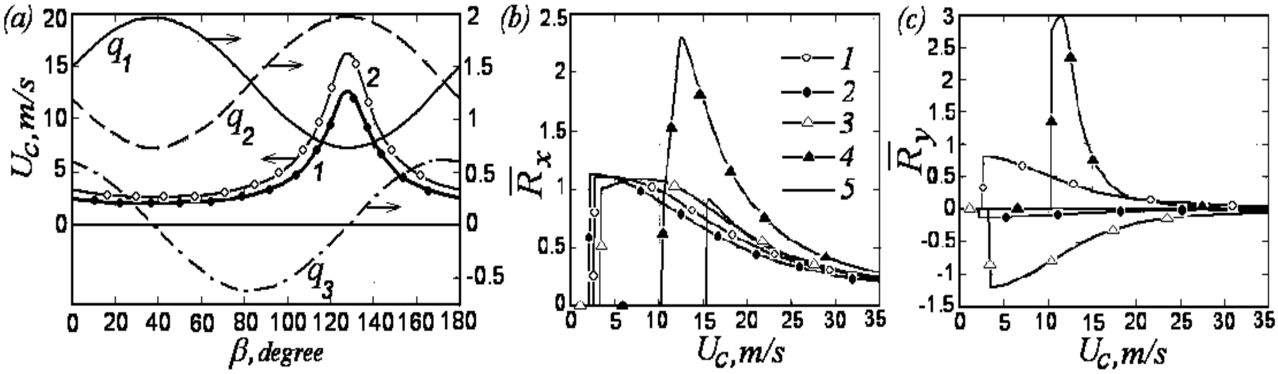

Figure. 1.

More detailed numerical results will be presented at the Workshop.

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